Research Statement

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1 Introduction

I am currently an assistant professor at Augusta State University. My dissertation advisor was Professor Qiudong Wang. My main research interest is in dynamical systems theory, and my thesis research is on the regularization of simultaneous binary collisions of the n-body problem. However, presently my focus is on the study of strange attractors.

2 Current Research

The periodically perturbed plane pendulum is a classical example of a periodically perturbed second order system [GH]. When a homoclinic solution is periodically perturbed, transversal intersections of stable and unstable manifolds occur within a certain range of forcing parameters, generating homoclinic tangles and strange attractors.

In [WO], it was proved that, for certain periodically forced second order systems, there exists a form of chaos that is less complicated in a range of parameters where the perturbed stable and unstable manifolds *do not* intersect. The idea is to start with a *non-resonant* and *dissipative* saddle that admits a homoclinic solution. The system is then subject to perturbation of the form

$$(-\mu(\rho h(x,y)+\sin\omega t), \mu(\rho h(x,y)+\sin\omega t))$$

where μ, ρ, ω are forcing parameters. The forced equations are written as an autonomous system in three dimensional space (x, y, θ), where x, y are phase variables, and θ is an angular variable representing time. A 2D Poincaré section in the space of (x, y, θ) is constructed, and the return maps induced by the three-dimensional autonomous flow are explicitly computed. The return maps obtained are shown to be admissible families of rank one maps. The existence of strange attractors with SRB measures then follows from a dynamics theory developed in recent years by Wang and Young ([WY1]-[WY3]).

2.1 The Periodically Perturbed Plane Pendulum

It appears that the analysis of [WO] could be applied to the study of a periodically forced pendulum

$$\frac{d^2\varphi}{dt^2} + \lambda\frac{d\varphi}{dt} + \sin\varphi = \gamma + \mu\sin\omega t \tag{1}$$

with weak constant torque, $\gamma > 0$, and damping, $\lambda > 0$.

We start with the second order system

$$\frac{d^2\varphi}{dt^2} + \lambda\frac{d\varphi}{dt} + \sin\varphi = \gamma \tag{2}$$

where parameters $\gamma, \lambda \geq 0$. Observe that system

$$\frac{d^2\varphi}{dt^2} + \sin\varphi = 0 \tag{3}$$

obtained from the (2) by disregarding weak constant torque and damping has a *homoclinic* orbit. However the system (3) is volume preserving and thus non-adequate for application of [WO] theory. Never the less, we use this fact to prove that, for the certain values of the parameters γ, λ, system (2) has a dissipative *homoclinic* saddle point. This observation enables us to apply the analysis of [WO] directly to (1) and show the existence of a new class of strange attractors in the periodically forced plane pendulum. Currently, we are in the process of writing out the details of the proofs.

3 Dissertation Research

Consider n particles moving in a 3-dimensional Euclidean space. Let m_k and $\mathbf{q}_k$ denote the mass and the position of the k-th particle, respectively. Assuming that interactions among particles are governed by the Newtonian Law of Gravitation, the motions of the particles are described by the n second-order differential equations,

$$m_k\frac{d^2\mathbf{q}_k}{dt^2} = \frac{\partial U}{\partial \mathbf{q}_k}, \quad k = 1, \dots, n \tag{4}$$

where U[1] is the potential function

$$U = \sum_{1\leq j<i\leq n} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}. \tag{5}$$

One of the main difficulties in the study of equation (4) is the existence of singularities of the potential function U. Let $r_{ij} = |\mathbf{q}_i - \mathbf{q}_j|$ be the distance between m_i and m_j, and let $r = \min_{1\leq i,j\leq n}\{r_{ij}\}$ be the minimum of all the mutual distances. We observe that $U \to \infty$ as $r \to 0$, and this happens when two or more particles collide in $\mathbb{R}^3$. We call a collision singularity involving two particles a *binary collision*,

[1]Note that in the physics literature $U = -\sum_{1\leq j<i\leq n} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}$.

one involving three particles a *triple collision*, and so on. It is also possible for more than one clusters of particles to collide at the same time. These are singularities of *simultaneous collisions*.

One way to deal with singularities of collisions is to introduce coordinate transformations to make these singularities *regular*. To be more precise, we say that a particular collision singularity is *regularizable* if we can find new phase variables and time so that the new vector field derived from equation (4) is *analytic* at places corresponding to this singularity. It is well-known that the singularity of binary collision is regularizable, and in fact there are many coordinate transformations that remove the singularity of binary collision.

For a long time, the regularizations of collision singularities remained an important theme in the studies of the n-body problem. Many influential developments, including Sundman's theory on power series solutions, Siegel's study on triple collisions, and the introduction of McGehee's transformations, are closely related to the studies on the regularizability of collision singularities. It turned out that collisions involving more than two particles are in general not regularizable, as was first proved by Siegel [S]. Consequently, the only singularity for which the issue of regularization is not yet settled is the singularity of *simultaneous binary collisions*. Despite much study ([Sa], [B], [SL], [E1], for instance) leaning towards an affirmative answer, no explicit regularization transformation was found for the singularity of simultaneous binary collisions.

The construction of regularization transformations for simultaneous binary collisions is the central theme of my dissertation research. For the first time, we are able to produce regularization transformations for simultaneous binary collisions in certain gravitational systems, including a pair of decoupled Kepler problems, a restricted four-body problem, and the collinear four-body problem. These cases are discussed separately in more detail in the following paragraphs.

3.1 Decoupled Kepler Problems

First, we constructed a coordinate transformation that removes the singularities of simultaneous binary collisions in a pair of decoupled Kepler problems. Suppose that we consider four gravitational particles $\{m_1, m_2, m_3, m_4\}$ moving on a line. We obtain a pair of decoupled Kepler problems by dropping the interactions between two mass groups $\{m_1, m_2\}$ and $\{m_3, m_4\}$.

The decoupled Kepler problems is the simplest among all the systems we considered. It consists of two independent two-body problems; each can be transformed into a harmonic oscillator by using a Levi-Civita transformation. However, in dealing with the singularities of simultaneous binary collisions, the two Kepler systems cannot be treated separately, since such treatment would create two new times. If one introduces a unified new time for regularization, then the ratio of the distances of the two colliding pairs appears persistently in the new equations, preventing the vector field obtained from becoming analytic at the point of collision. This is the main reason why a change of coordinates that regularizes the singularity of simultaneous binary collision was not found, even though the system studied here is integrable.

The first part of my thesis is an explicit construction of a coordinate transformation that regularizes the singularity of simultaneous binary collisions for the decoupled Kepler systems. We introduced the differences of colliding times of the different pairs for the surrounding solutions as a new phase variable, which we denote as Y. It turned out that the ratio of the colliding distances can be written in a form that is analytic in terms of Y and other new phase variables at the places for simultaneous binary collisions. To the best of our knowledge, this coordinate transformation is the first ever constructed that regularizes the singularity of simultaneous binary collision in any system governed by the Newtonian law of gravitation.

3.2 A Restricted Four-Body Problem

The next part of my thesis was to extend the result obtained above to gravitational systems in which interactions between colliding pairs do exist. We considered again the motion of four mass particles $\{m_1, m_2, m_3, m_4\}$ on a line and let $m_1 = m_4 = 0$. Unlike the decoupled Kepler problems considered earlier, this restricted four-body problem is not integrable. The definition of the new variable Y, which was previously straightforward using functions explicitly derived from integrals of energy, becomes analytically implicit. Consequently, a one-step definition is replaced by a long and complicated inductive process. The induction was carried out in detail in my dissertation. The issue of convergence of all the functions formally derived from this complicated inductive process is also carefully investigated. At the end, a coordinate transform that regularizes the singularity of simultaneous binary collisions is constructed. Instead of defining Y in closed formulas, the new variable Y is now defined explicitly by a convergent inductive process.

A paper on the results as described in the above (Sects. 3.1 and 3.2) has been submitted for publication [PW].

3.3 Collinear Four-Body Problem

The next step was to generalize the results obtained above to the collinear four-body problem without assuming $m_1 = m_4 = 0$. Here we unfortunately ran into seriously technical obstacles in obtaining a well-defined inductive process for the new variable Y. In order to write the ratio of colliding distances in analytic terms at the places of simultaneous binary collisions, we were forced to take a new route. In this case, our construction of a desired coordinate transformation involves a study of the collision manifold for simultaneous binary collisions. It turned out that, through an explicit computation of the stable manifold of the collision point, the ratio of colliding distances can be written as a convergent power series of new phase variables at the places of collision. The computational process involved is long and tedious.

Up to this point we have finished writing the first draft of this result.

3.4 General Collision Manifold

The final part of my thesis is a by-product of the studies described in Sect. 3.3. Based on McGehee's transformation, boundary manifolds for general collisions have been constructed previously by others in a rather tedious way [E2]. Based on a coordinate transformation introduced by Wang [W], we were able to construct boundary manifolds for general collisions and recover all that was proved in [E2] in only a few pages. This is now leading us to a study on the general nature of solutions approaching singularities of collision. The aim of this investigation is to give a systematic treatment that would enable us to quickly recover many of the results about collision singularities previously proved by Don Saari ([PS], [SH]) and others [E2].

We will include the results of this investigation in my dissertation as an appendix.

4 Future Research

While I am still extending my knowledge base into the studies of strange attractors, there are a few problems directly inspired by my dissertation research that I would like to work on.

(A) Geometric Regularization My first problem is to see whether the result on analytic regularization we obtained so far could be used to improve the existing results on *geometric* regularizations for the collinear four-body problem. I have invested some time this summer on this problem, and it appears that there are many interesting details to figure out. Despite many previous efforts ([SL], [E3]), the progress made in this direction has been very limited so far.

(B) Planar and Spatial Four-Body Problem It appears feasible that our technique for regularization could be extended to treat planar and spatial systems. Some obstacles to such extensions are well-known in the literature. For instance instead of Levi-Civita transformation one would have to use Kustaanheimo-Stiefel transformations for the spatial problem.

The following question is also on the study of the n-body problem.

(C) Global Phase Structure of the Collinear Four-Body Problem It seems natural for us to follow the studies of Meyer and Wang in [MW1] to obtain a sketch of the global phase portrait for the collinear four-body problem. We would try to first construct a symbolic coding and to prove that there are solutions corresponding to all or the majority of the proposed codings. Results in this direction have been produced in the past for the collinear three-body problem ([MW1]) and the restricted isosceles three body problem ([MW2]). We could first introduce certain symmetries to reduce the degrees of freedom of the problem at hand, then try to extend the results obtained to less restrictive settings.

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