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Asymptotically minimal uncertainty states for time-dependent oscillators

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We consider the time-dependent Schrödinger equation in one spatial dimension with a time-dependent quadratic Hamiltonian and, under appropriate assumptions on the coefficient functions in the Hamiltonian, construct solutions that approach minimal uncertainty states for large times. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4773874]

I. INTRODUCTION

In this paper we consider certain solutions $\Psi(x, t)$ of time-dependent Schrödinger equations of the form

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} = &- \hbar^2\alpha(t)\frac{\partial^2\Psi}{\partial x^2} - i\hbar\beta(t)\left(x\frac{\partial\Psi}{\partial x} + \frac{\partial}{\partial x}(x\Psi)\right) \\ &+ \gamma(t)x^2\Psi - i\hbar\delta(t)\frac{\partial\Psi}{\partial x} + \varepsilon(t)x\Psi + \zeta(t)\Psi, \end{aligned} \tag{1}$$

where α, β, γ, δ, ε, and ζ are real continuous functions. We shall focus on only the $\beta(t) \equiv 0$ case though we maintain the presence of the "cross terms" in this introduction for the sake of generality. Our main interest lies in the uncertainties Δx and Δp in position and momentum, respectively, associated with the state $\Psi(x, t)$.

Given a solution $\Psi(x, t) \in L^2(\mathbb{R}, dx)$ for $t \geq 0$ normalized so that

$$\int_{-\infty}^{\infty} |\Psi(x, t)|^2 \, dx = 1,$$

the expectations $\langle x\rangle$ and $\langle p\rangle$ of position and momentum are given by

$$\langle x\rangle = \int_{-\infty}^{\infty} \overline{\Psi(x, t)} x \Psi(x, t)\, dx$$

and

$$\langle p\rangle = \int_{-\infty}^{\infty} \overline{\Psi(x, t)}(-i\hbar\frac{\partial}{\partial x})\Psi(x, t)\, dx$$

and the uncertainties in these quantities by

$$(\Delta x)^2 = \int_{-\infty}^{\infty} \overline{\Psi(x, t)}(x - \langle x\rangle)^2\Psi(x, t)\, dx$$

and

$$(\Delta p)^2 = \int_{-\infty}^{\infty} \overline{\Psi(x, t)}(-i\hbar\frac{\partial}{\partial x} - \langle p\rangle)^2\Psi(x, t)\, dx.$$

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The solutions of (1) we study are constructed using Hagedorn 's[1] "semi-classical" wave packet states $\{\varphi_n\}$.

Definition 1.1: For $n \in \mathbb{Z}^+ \cup \{0\}$, $a, \eta \in \mathbb{R}$, $\hbar > 0$, *and* $A, B \in \mathbb{C}$ *satisfying*

$$A\overline{B} + \overline{A}B = 2 \tag{2}$$

we let

$$\varphi_n(A, B, \hbar, a, \eta, x) = 2^{-n/2}(n!)^{-1/2}(\pi\hbar)^{-1/4}A^{-(n+1)/2}\overline{A}^{n/2}H_n(\hbar^{-1/2}|A|^{-1}(x-a)) \times \exp\left\{-\frac{1}{2\hbar}BA^{-1}(x-a)^2 + \frac{i}{\hbar}\eta(x-a)\right\},$$

where H_n *is the nth order Hermite polynomial.*

The condition (2) is not really restrictive in the sense that any complex number Γ with $\mathrm{Re}(\Gamma) > 0$ can be written in the form $\Gamma = BA^{-1}$ for some A and B satisfying (2). The purpose of writing $\Gamma = BA^{-1}$ with A and B satisfying (2) is to insure that

$$\mathrm{Re}(BA^{-1}) = |A|^{-2} \quad \text{and} \quad \mathrm{Re}(AB^{-1}) = |B|^{-2},$$

which in turn implies that the state $\varphi_n(A, B, \hbar, a, \eta, x)$ is normalized in $L^2(\mathbb{R}, dx)$ and has expected position and momentum

$$\langle x \rangle = a \quad \text{and} \quad \langle p \rangle = \eta$$

and uncertainties

$$\Delta x = \sqrt{\hbar\left(n + \frac{1}{2}\right)}\,|A| \quad \text{and} \quad \Delta p = \sqrt{\hbar\left(n + \frac{1}{2}\right)}\,|B|.$$

We note that if A and B satisfy (2) then so do $e^{i\theta}A$ and $e^{i\theta}B$ for any $\theta \in \mathbb{R}$, and that it also follows from condition (2) that $|A||B| \geq 1$ (this is the Heisenberg uncertainty principle).

The following proposition[2] can be proved easily by substitution and induction.

Proposition 1.1: Suppose $a_0, \eta_0 \in \mathbb{R}$ *and* $A_0, B_0 \in \mathbb{C}$ *are given and such that* $A_0\overline{B_0} + \overline{A_0}B_0 = 2$. *Let* $(a(t), \eta(t), A(t), B(t), S(t))$ *be the unique solution of the system of ordinary differential equations*

$$\dot{a} = 2\alpha(t)\eta + 2\beta(t)a + \delta(t), \tag{3}$$

$$\dot{\eta} = -2\beta(t)\eta - 2\gamma(t)a - \varepsilon(t), \tag{4}$$

$$\dot{A} = 2i\alpha(t)B + 2\beta(t)A, \tag{5}$$

$$\dot{B} = -2\beta(t)B + 2i\gamma(t)A, \tag{6}$$

$$\dot{S} = \alpha(t)\eta^2 - \gamma(t)a^2 - \varepsilon(t)a - \zeta(t), \tag{7}$$

subject to the initial condition $(a(0), \eta(0), A(0), B(0), S(0)) = (a_0, \eta_0, A_0, B_0, 0)$. *Then*

$$\Psi(x, t) = e^{iS(t)/\hbar}\varphi_n\left(A(t), B(t), \hbar, a(t), \eta(t), x\right) \tag{8}$$

is a solution of the time-dependent Schrödinger equation (1).

Note that Eqs. (3) and (4) are the classical equations of motion for the position a and momentum η with Hamiltonian

$$H(a, \eta, t) = \alpha(t)\eta^2 + 2\beta(t)a\eta + \gamma(t)a^2 + \delta(t)\eta + \varepsilon(t)a + \zeta(t)$$

and that (7) is the associated classical action. The wave function (8) evolves in time with expected position and momentum concentrated on the classical values. For completeness we remark that the branch of the square root in (8) is determined by continuity in t.

Equations (5) and (6) preserve condition (2), that is, $A(t)\overline{B(t)}+\overline{A(t)}B(t)=2$ for $t>0$. The spreadings (uncertainties) in position and momentum space are determined by $|A(t)|$ and $|B(t)|$, respectively. We further remark here that differentiating (3) and (4) with respect to the initial conditions a_0, η_0 and comparing with (5) and (6) reveals that

$$A(t)=A_0\frac{\partial a}{\partial a_0}+iB_0\frac{\partial a}{\partial \eta_0}$$

$$B(t)=B_0\frac{\partial \eta}{\partial \eta_0}-iA_0\frac{\partial \eta}{\partial a_0}$$

and furthermore

$$U(t)=\begin{bmatrix}\frac{\partial a}{\partial a_0} & \frac{\partial a}{\partial \eta_0}\\ \frac{\partial \eta}{\partial a_0} & \frac{\partial \eta}{\partial \eta_0}\end{bmatrix}$$

is a fundamental matrix for the system (5) and (6), and for the homogeneous ($\delta(t)=\varepsilon(t)=0$) version of (3) and (4). Condition (2) is then seen to be a restatement of the fact that

$$\det U(t)=\frac{\partial a}{\partial a_0}\frac{\partial \eta}{\partial \eta_0}-\frac{\partial a}{\partial \eta_0}\frac{\partial \eta}{\partial a_0}=1.$$

A. Minimum uncertainty states

We first consider the possibility that the solutions constructed in the proposition above are minimal uncertainty states, that is, the possibility that they satisfy $\Delta x\Delta p=$ constant. Let

$$\Gamma(t)=\frac{B(t)}{A(t)}=\rho(t)+i\sigma(t)$$

and note that $\Gamma(t)$ satisfies the Riccati equation

$$\dot{\Gamma}=2i\gamma(t)-4\beta(t)\Gamma-2i\alpha(t)\Gamma^2, \tag{9}$$

while its real and imaginary parts satisfy the system

$$\dot{\rho}=-4\beta(t)\rho+4\alpha(t)\sigma\rho, \tag{10}$$

$$\dot{\sigma}=2\gamma(t)-2\alpha(t)\rho^2-4\beta(t)\sigma+2\alpha(t)\sigma^2. \tag{11}$$

If we suppose that the product of uncertainties $\Delta x\Delta p$ is constant, then differentiating $|A(t)|^2|B(t)|^2$ and using (5) and (6) leads to

$$\sigma(t)\left(\alpha(t)\,|\Gamma(t)|^2-\gamma(t)\right)=0 \tag{12}$$

and it is not difficult to see using (9) or (10) and (11) that a necessary condition for solutions consistent with (12) is

$$\sqrt{|\alpha(t)|}\left(8\alpha(t)\beta(t)\gamma(t)+\alpha(t)\gamma'(t)-\gamma(t)\alpha'(t)\right)=8c\alpha(t)^2\gamma(t)\sqrt{|\gamma(t)|}, \tag{13}$$

for some constant $c\in[-1,1]$.

The condition (13) is quite restrictive. However, there are interesting Hamiltonians (other than the trivial case where α and γ are constant and $\beta=c=0$) for which it holds, for example, the *Caldirola*[3]*-Kanai*[4] Hamiltonian

$$H(a,\eta,t)=\frac{1}{2m_0}e^{-2\lambda t}\eta^2+\frac{1}{2}m_0\omega^2e^{2\lambda t}a^2$$

in the "underdamped" case $\lambda<\omega$ with $c=\lambda/\omega$. A detailed analysis of minimum uncertainty states for this Hamiltonian can be found in the work of Kim,[5] but we note that our solutions (8) are, with a proper choice of initial conditions for $A(t)$ and $B(t)$, minimal uncertainty states in this case. For some other well-known systems, such as an abundance of time dependent harmonic oscillators (see Predrosa[6] and the references cited therein) including, as a example, the weakening spring

$$H(a, \eta, t) = \frac{1}{2m}\eta^2 + \frac{1}{2}\frac{k}{1+\lambda t}a^2, \tag{14}$$

or the lengthening simple pendulum[7]

$$H(a, \eta, t) = \frac{1}{2m(l+kt)^2}\eta^2 - mg(l+kt)(1-\frac{1}{2}a^2), \tag{15}$$

we see that condition (13) is violated and therefore our solutions (8) in these cases cannot be minimum uncertainty states.

B. Asymptotically minimal uncertainty states

Though in general the solutions given by (8) are not minimum uncertainty states we can show that, for a large class of Hamiltonians, by a suitable choice of initial conditions our solutions asymptotically approach minimum uncertainty states, that is, we will show that there is a choice of suitable initial conditions $A(0) = A_0$ and $B(0) = B_0$ that leads to $\lim_{t\to\infty}|A(t)B(t)| = 1$. For all of the examples mentioned above (harmonic oscillator, Caldirola-Kanai, lengthening simple pendulum, weakening spring) this can be done by explicitly solving the system for A and B, which we will illustrate for "Bessel-like" Hamiltonians in Sec. II. In Sec. III we prove a more general result that includes these Hamiltonians as a special case.

II. SOME PRELIMINARY RESULTS

In the remainder of this paper we consider only Hamiltonians with no "cross terms,"that is, Hamiltonians of the form

$$H(a, \eta, t) = \alpha(t)\eta^2 + \gamma(t)a^2 + \delta(t)\eta + \varepsilon(t)a + \zeta(t),$$

with reasonable assumptions on the coefficient functions. We can further ignore the $\zeta(t)$ term if we like as it plays no role in the classical mechanics (3) and (4) or spreadings (5) and (6) and enters the quantum mechanics only through a phase. Prior to stating and proving our main result we consider two physically important special cases.

A. Hamiltonians of Bessel type

Consider the Hamiltonian

$$H(a, \eta, t) = \kappa(\lambda t+1)^{2q-1}\eta^2 + \frac{\lambda^2\left(b^2c^2(\lambda t+1)^{2c} - c^2p^2 + q^2\right)}{4\kappa(\lambda t+1)^{2q+1}}a^2 + \zeta(t) \tag{16}$$

for some real function $\zeta(t)$ and real constants $\kappa > 0$, $c > 0$, $\lambda > 0$, $b \geq 0$, $p \geq 0$, and q. Notice that for $q = 1/2$, $\kappa = 1/(2m)$, $p = 1$, $c = 1/2$, $b = 2\lambda^{-1}\sqrt{k/m}$, and $\zeta(t) = 0$ this is the Hamiltonian for the weakening spring (14) while for $q = -1/2$, $\kappa = 1/(2ml^2)$, $p = 1$, $c = 1/2$, $b = 2k^{-1}\sqrt{gl}$, $\lambda = k/l$, and $\zeta(t) = -mg(l + kt)$ this is the Hamiltonian for the lengthening simple pendulum (15).

The system for $A(t)$ and $B(t)$ for Hamiltonians of the form (16) is

$$\dot{A} = 2i\kappa(\lambda t+1)^{2q-1}B, \tag{17}$$

$$\dot{B} = i\frac{\lambda^2\left(b^2c^2(\lambda t+1)^{2c} - c^2p^2 + q^2\right)}{2\kappa(\lambda t+1)^{2q+1}}A, \tag{18}$$

for $t \geq 0$ which, upon differentiating the first, substituting the second, and using the first again to eliminate B results in the second order ordinary differential equation

$$\ddot{A} - \frac{(2q-1)\lambda}{\lambda t+1}\dot{A} + \lambda^2\left(\frac{q^2-c^2p^2}{(\lambda t+1)^2} + b^2c^2(\lambda t+1)^{2c-2}\right)A = 0.$$

Changing variable to $\tau = \lambda t + 1$ gives a well-known[8] general Bessel equation for $A(\tau)$

$$\frac{d^2A}{d\tau^2} + \frac{1-2q}{\tau}\frac{dA}{d\tau} + \left(\frac{q^2 - c^2p^2}{\tau^2} + b^2c^2\tau^{2c-2}\right)A = 0,$$

for $\tau \geq 1$ having solution

$$A(\tau) = \tau^q \left(c_1 Y_p\left(b\tau^c\right) + c_2 J_p\left(b\tau^c\right)\right),$$

for some choice of constants c_1 and c_2 depending on the initial conditions. Using (17) we obtain

$$B(\tau) = \frac{i\lambda\tau^{-q}}{2\kappa}\left(c_2\left(bc\tau^c J_{p+1}\left(b\tau^c\right) - (cp+q)J_p\left(b\tau^c\right)\right)\right.$$
$$\left. + c_1\left((3cp-q)Y_p\left(b\tau^c\right) + bc\tau^c Y_{p+1}\left(b\tau^c\right)\right)\right).$$

It is then easy to show using the large z asymptotics of the Bessel functions,[9] namely,

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\cos(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + \mathcal{O}(z^{-3/2})$$
$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}}\sin(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + \mathcal{O}(z^{-3/2})$$

that

$$|A(\tau)B(\tau)|^2 = \frac{c^2\lambda^2\left|c_1\sin(\omega_p) + c_2\cos(\omega_p)\right|^2\left|c_1\sin(\omega_{p+1}) + c_2\cos(\omega_{p+1})\right|^2}{\pi^2\kappa^2} + \mathcal{O}(\tau^{-c})$$

as $\tau \to \infty$ where

$$\omega_\nu = b\tau^c - \frac{1}{2}\nu\pi - \frac{1}{4}\pi.$$

Clearly then the choice $c_2 = -ic_1$ leads to a solution for which

$$|A(\tau)B(\tau)| = \frac{c\lambda}{\pi\kappa}|c_1|^2 + \mathcal{O}(\tau^{-c}).$$

Now, since $t = 0$ corresponds to $\tau = 1$, we see that, with $c_2 = -ic_1$,

$$A(1)\overline{B(1)} + \overline{A(1)}B(1) = \frac{2c\lambda}{\pi\kappa}|c_1|^2,$$

hence if we choose $c_1 = \sqrt{\pi\kappa/(c\lambda)}e^{i\theta}$ for arbitrary $\theta \in [0, 2\pi)$ we have proven the following result:

Proposition 2.1: There are complex numbers A_0 and B_0 satisfying $A_0\overline{B_0} + \overline{A_0}B_0 = 2$ such that the solution of the system (17) and (18) with initial conditions $A(0) = A_0$ and $B(0) = B_0$ satisfies $|A(t)B(t)| = 1 + \mathcal{O}(t^{-c})$ as $t \to \infty$.

As an example of the explicitness of the construction above we consider the lengthening pendulum Hamiltonian (15). The argument above shows that if

$$A_0 = -i\sqrt{\frac{\pi}{kml}}e^{i\phi}H_1^{(1)}\left(\frac{2\sqrt{gl}}{k}\right),$$
$$B_0 = l\sqrt{\frac{\pi gm}{k}}e^{i\phi}H_2^{(1)}\left(\frac{2\sqrt{gl}}{k}\right),$$

where $H_n^{(1)}(z)$ denotes the Hankel function of the first kind and $\phi \in \mathbb{R}$ is arbitrary then

$$A_0\overline{B_0} + \overline{A_0}B_0 = 2$$

and the solution of

$$\dot{A} = \frac{i}{m(l+kt)^2} B,$$
$$\dot{B} = img(l+kt)A,$$

satisfies

$$1 \leq |A(t)B(t)| \leq 1 + \mathcal{O}(\frac{1}{\sqrt{l+kt}})$$

for $t \geq 0$.

B. Hamiltonians asymptotic to harmonic oscillators

In this section we establish the existence of asymptotically minimal uncertainty states along with the leading asymptotics of the uncertainties Δx and Δp for Hamiltonians of the form

$$H(a, \eta, t) = \alpha(t)\eta^2 + \gamma(t)a^2 + \delta(t)\eta + \varepsilon(t)a,$$

where $\alpha(t)$ and $\gamma(t)$ have finite, nonzero limits as $t \to \infty$, and $\delta(t)$and $\varepsilon(t)$ vanish as $t \to \infty$. In other words, the Hamiltonian approaches that of a simple harmonic oscillator. The existence of asymptotically minimal uncertainty states in this setting is surely plausible, yet this class is not large enough to contain many Hamiltonians of interest. However, this simple exercise is not only (at least mildly) physically interesting in itself but is the key for dealing with the more general case in Sec. III.

For the rest of this section we require the following technical assumptions on the coefficient functions:

1. $\alpha(t)$ and $\gamma(t)$ are positive and twice continuously differentiable;
2. $\lim_{t\to\infty}\alpha(t) = \alpha_\infty > 0$ and $\lim_{t\to\infty}\gamma(t) = \gamma_\infty > 0$ with $|\alpha(t) - \alpha_\infty| = \mathcal{O}(1/t)$ and $|\gamma(t) - \gamma_\infty| = \mathcal{O}(1/t)$ as $t \to \infty$;
3. $\delta(t)$ and $\varepsilon(t)$ are continuous and along with $\dot{\alpha}(t)$, $\dot{\gamma}(t)$, $\ddot{\alpha}(t)$, and $\ddot{\gamma}(t)$ are all $\mathcal{O}(1/t^{1+\mu})$ as $t \to \infty$ for some $\mu > 0$.

These assumptions are not optimal (for example, continuity can be replaced by piecewise continuity) but they are general enough for our purpose.

We start by establishing the boundedness of the classical motion.

Lemma 2.1: Under the assumptions on α, γ, δ, and ε above, all solutions of the classical motion

$$\dot{a} = 2\alpha(t)\eta + \delta(t)$$
$$\dot{\eta} = -2\gamma(t)a - \varepsilon(t)$$

remain bounded, that is, for any initial condition (a_0, η_0) there is a constant $C(a_0, \eta_0)$ such that

$$\max\{|a(a_0, \eta_0, t)|\,, |\eta(a_0, \eta_0, t)|\} < C(a_0, \eta_0).$$

Proof: The proof is straightforward using Sec. V.11.1 of Courant and Hilbert[10] for the homogeneous version of the system and then variation of parameters for the full system. □

We can now state and prove the main result of this section.

Proposition 2.2: Under the same assumptions as in the lemma, let

$$\phi(t) = 2\int_0^t \sqrt{\alpha(s)\gamma(s)}ds.$$

Then, there is a choice of initial conditions A_0 and B_0 with $A_0\overline{B_0}+\overline{A_0}B_0=2$ and such that the asymptotic behavior of the solution of the system

$$\dot{A}=2i\alpha(t)B$$
$$\dot{B}=2i\gamma(t)A$$

is

$$A(t)=e^{i\phi(t)}\left(\frac{\alpha(t)}{\gamma(t)}\right)^{1/4}+\mathcal{O}(\frac{1}{t^{\mu}})$$
$$B(t)=e^{i\phi(t)}\left(\frac{\gamma(t)}{\alpha(t)}\right)^{1/4}+\mathcal{O}(\frac{1}{t^{\mu}})$$

as $t\to\infty$ and, in particular,

$$|A(t)B(t)|=1+\mathcal{O}(\frac{1}{t^{\mu}})$$

as $t\to\infty$.

Proof: To prove the proposition we consider the quantities

$$\sigma(t)=A(t)-e^{i\phi(t)}\left(\frac{\alpha(t)}{\gamma(t)}\right)^{1/4},$$
$$\omega(t)=B(t)-e^{i\phi(t)}\left(\frac{\gamma(t)}{\alpha(t)}\right)^{1/4},$$

which satisfy

$$\dot{\sigma}(t)=2i\alpha(t)\omega(t)+e^{i\phi(t)}\frac{\alpha(t)\dot{\gamma}(t)-\gamma(t)\dot{\alpha}(t)}{4\alpha(t)^{3/4}\gamma(t)^{5/4}}$$
$$\dot{\omega}(t)=2i\gamma(t)\sigma(t)-e^{i\phi(t)}\frac{\alpha(t)\dot{\gamma}(t)-\gamma(t)\dot{\alpha}(t)}{4\alpha(t)^{5/4}\gamma(t)^{3/4}}$$

or,

$$\frac{d}{dt}\begin{bmatrix}\sigma\\ i\omega\end{bmatrix}=2\begin{bmatrix}0&\alpha(t)\\ -\gamma(t)&0\end{bmatrix}\begin{bmatrix}\sigma\\ i\omega\end{bmatrix}+e^{i\phi(t)}\frac{\alpha(t)\dot{\gamma}(t)-\gamma(t)\dot{\alpha}(t)}{4\alpha(t)^{3/4}\gamma(t)^{3/4}}\begin{bmatrix}\gamma(t)^{-1/2}\\ -i\alpha(t)^{-1/2}\end{bmatrix}.$$

Variation of parameters then gives the solution

$$\begin{bmatrix}\sigma(t)\\ i\omega(t)\end{bmatrix}=U(t)\begin{bmatrix}\sigma(0)\\ i\omega(0)\end{bmatrix}+\int_0^t U(t-s)e^{i\phi(s)}\frac{\alpha(s)\dot{\gamma}(s)-\gamma(s)\dot{\alpha}(s)}{4\alpha(s)^{3/4}\gamma(s)^{3/4}}\begin{bmatrix}\gamma(s)^{-1/2}\\ -i\alpha(s)^{-1/2}\end{bmatrix}ds,$$

where $U(t)$ is the fundamental matrix given in the remarks after Proposition 1.1. Consider for the moment the integral

$$I(t)=\int_0^t U(s)^{-1}e^{i\phi(s)}\frac{\alpha(s)\dot{\gamma}(s)-\gamma(s)\dot{\alpha}(s)}{4\alpha(s)^{3/4}\gamma(s)^{3/4}}\begin{bmatrix}\gamma(s)^{-1/2}\\ -i\alpha(s)^{-1/2}\end{bmatrix}ds$$

and note that (where $\|\cdot\|$ denotes both any vector norm and the induced matrix norm)

$$\|I(t)\|\le\int_0^t\left\|U(s)^{-1}\right\|\left|\frac{\alpha(s)\dot{\gamma}(s)-\gamma(s)\dot{\alpha}(s)}{4\alpha(s)^{3/4}\gamma(s)^{3/4}}\right|\left\|\begin{bmatrix}\gamma(s)^{-1/2}\\ -i\alpha(s)^{-1/2}\end{bmatrix}\right\|ds.$$

By the lemma above the norm of the inverse of the propagator $U(t)$ is bounded by some constant (its determinant is constant and its elements are therefore just rearrangements of solutions of the classical motion). The integrand is then $\mathcal{O}(1/s^{1+\mu})$ by our assumptions and therefore the integral converges as $t\to\infty$. If we let

$$\begin{bmatrix}\sigma(0)\\ i\omega(0)\end{bmatrix}=-\int_0^\infty U(s)^{-1}e^{i\phi(s)}\frac{\alpha(s)\dot{\gamma}(s)-\gamma(s)\dot{\alpha}(s)}{4\alpha(s)^{3/4}\gamma(s)^{3/4}}\begin{bmatrix}\gamma(s)^{-1/2}\\ -i\alpha(s)^{-1/2}\end{bmatrix}ds, \tag{19}$$

then

$$\begin{bmatrix} \sigma(t) \\ i\omega(t) \end{bmatrix} = -\int_t^\infty U(s-t)^{-1} e^{i\phi(s)} \frac{\alpha(s)\dot{\gamma}(s) - \gamma(s)\dot{\alpha}(s)}{4\alpha(s)^{3/4}\gamma(s)^{3/4}} \begin{bmatrix} \gamma(s)^{-1/2} \\ -i\alpha(s)^{-1/2} \end{bmatrix} ds$$

and

$$\left\| \begin{bmatrix} \sigma(t) \\ i\omega(t) \end{bmatrix} \right\| = \mathcal{O}(\frac{1}{t^\mu}).$$

This proves the asymptotics of $A(t)$ and $B(t)$. It only remains to show that the initial conditions A_0 and B_0 induced by (19) satisfy $A_0\overline{B_0} + \overline{A_0}B_0 = 2$. To this end we first note that, from the definitions of $\sigma(t)$ and $\omega(t)$,

$$\begin{aligned} A(t)\overline{B(t)} + \overline{A(t)}B(t) = 2 + \omega(t)\overline{\sigma(t)} + \sigma(t)\overline{\omega(t)} \\ + \left(\frac{\gamma(t)}{\alpha(t)}\right)^{1/4} \left(\overline{\sigma(t)}e^{i\phi(t)} + \sigma(t)e^{-i\phi(t)}\right) \\ + \left(\frac{\alpha(t)}{\gamma(t)}\right)^{1/4} \left(\overline{\omega(t)}e^{i\phi(t)} + \omega(t)e^{-i\phi(t)}\right). \end{aligned}$$

We know that the quantity on the left side is conserved. As $t \to \infty$, the asymptotics established above and our hypotheses show that the right side approaches 2. This completes the proof of the proposition. □

We remark that although in general the initial conditions generated by (19) will not be explicitly computable, this relation is numerically tractable, that is, we can at least recover the sought-after A_0 and B_0 for specific examples by numerically solving the system for $U(t)$ over an appropriately long time.

III. THE MAIN RESULT

We now establish the existence of asymptotically minimal uncertainty states along with the leading asymptotics of the uncertainties Δx and Δp for Hamiltonians of the form

$$H(a, \eta, t) = \alpha(t)\eta^2 + \gamma(t)a^2 + \delta(t)\eta + \varepsilon(t)a,$$

where α and γ are positive and at least five times continuously differentiable functions satisfying some further technical conditions we make clear in the actual statement of the theorem. The main idea of this section is that a suitable change of variables reduces this case to that studied in Sec. II B. Though we sacrifice some explicitness for the sake of generality the result below extends those in that section although our method of proof requires some (removable) restrictions on the parameters in the case of the Hamiltonian (16).

We now state our main result.

Proposition 3.1: Suppose that $q(t)$ defined by

$$q(t) = 4\alpha\gamma - \frac{1}{4}\left(\frac{\dot{\alpha}}{\alpha}\right)^2 + \frac{1}{2}\frac{d}{dt}\left(\frac{\dot{\alpha}}{\alpha}\right)$$

is strictly positive and that $F(\tau)$ defined by

$$F(\tau) = -q(\tau)^{-1/4}\frac{d^2}{d\tau^2}\left(q(\tau)^{-1/4}\right),$$

where

$$\tau = \int_0^t q(s)^{1/2} ds$$

is continuously differentiable and satisfies

$$|F(\tau)| < \frac{\lambda}{\nu\tau^{\nu}}, \quad |F'(\tau)| < \frac{\lambda}{\tau^{1+\nu}},$$

for some positive constants λ *and* ν*. Suppose further that*

$$\lim_{t\to\infty} \frac{2q(t)\dot{\alpha}(t) - \alpha(t)\dot{q}(t)}{q(t)^{3/2}\alpha(t)} = 0. \tag{20}$$

Then there exist complex numbers A_0 *and* B_0 *satisfying* $A_0\overline{B_0} + \overline{A_0}B_0 = 2$ *such that the solutions*

$$\Psi(x,t) = e^{iS(t)/\hbar}\varphi_n(A(t), B(t), \hbar, a(t), \eta(t), x), \ \ n = 0, 1, 2, \ldots$$

of the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\hbar^2\alpha(t)\frac{\partial^2\Psi}{\partial x^2} + \gamma(t)x^2\Psi - i\hbar\delta(t)\frac{\partial\Psi}{\partial x} + \varepsilon(t)x\Psi$$

constructed using Proposition 1.1 are asymptotically minimal uncertainty states, that is, such that

$$|\Delta x\Delta p| = \hbar\left(n+\frac{1}{2}\right)|A(t)B(t)| \sim \hbar\left(n+\frac{1}{2}\right) \text{ as } t\to\infty.$$

Moreover, the asymptotics of the spreadings $A(t)$ *and* $B(t)$ *are given by*

$$A(t) \sim \frac{\sqrt{2\alpha(t)}}{q(t)^{1/4}} e^{i\int_0^t \sqrt{q(s)}ds}$$

and

$$B(t) \sim \left(\frac{q(t)^{1/4}}{\sqrt{2\alpha(t)}} - \frac{i}{4\sqrt{2}}\frac{2q(t)\dot{\alpha}(t) - \alpha(t)\dot{q}(t)}{q(t)^{5/4}\alpha(t)^{3/2}}\right) e^{i\int_0^t \sqrt{q(s)}ds}$$

as $t \to \infty$.

The remainder of this section is devoted to the proof of this proposition. The system

$$\dot{A} = 2i\alpha(t)B \tag{21}$$

$$\dot{B} = 2i\gamma(t)A \tag{22}$$

reduces to a single second order ODE

$$\ddot{A} - \frac{\dot{\alpha}(t)}{\alpha(t)}\dot{A} + 4\alpha(t)\gamma(t)A = 0. \tag{23}$$

Recall that $\alpha(t) > 0$ and let

$$A(t) = cz(t)\sqrt{\alpha(t)}, \tag{24}$$

where c is a constant to be specified later. Then (23) becomes

$$\ddot{z} + q(t)z = 0, \tag{25}$$

where

$$q(t) = 4\alpha\gamma - \frac{1}{4}\left(\frac{\dot{\alpha}}{\alpha}\right)^2 + \frac{1}{2}\frac{d}{dt}\left(\frac{\dot{\alpha}}{\alpha}\right).$$

We assume $q(t) \geq 0$ and, following Liouville,[11] we now introduce the new change of variables in (25)

$$\tau = \int_0^t q(s)^{1/2}ds, \quad w(\tau) = q(\tau)^{1/4}z(\tau),$$

which reduces Eq. (25) to

$$w'' + (1 + F(\tau))\,w = 0, \tag{26}$$

where differentiation is with respect to time τ now and the function $F(\tau)$ is given as

$$F(\tau) = -q(\tau)^{-1/4}\frac{d^2}{d\tau^2}\left(q(\tau)^{-1/4}\right).$$

Let $w' = iv$ then Eq. (26) in new coordinates is given as the system,

$$\frac{d}{d\tau}\begin{bmatrix} w \\ iv \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(1+F(\tau)) & 0 \end{bmatrix}\begin{bmatrix} w \\ iv \end{bmatrix} \tag{27}$$

which we also write as

$$w' = iv,$$

$$v' = i\,(1+F(\tau))\,w,$$

for the sake of comparison with (21) and (22). We now study the asymptotic behavior of the system (27) following the approach in Proposition 2.2. We make the following assumption on the function $F(\tau)$:

Hypothesis 3.1: Suppose that $F(\tau) \in \mathbf{C}^1$ such that

$$|F(\tau)| < \frac{\lambda}{\nu\tau^{\nu}}, \quad |F'(\tau)| < \frac{\lambda}{\tau^{1+\nu}}, \tag{28}$$

where λ and ν are positive constants.

We now state a simple lemma and its corollaries which we prove in the Appendix.

Lemma 3.1: Suppose that (26) satisfies Hypothesis 3.1 then every solution of Eq. (26) is bounded for $\tau \to \infty$.

Let $U(\tau)$ be *fundamental matrix solution* of the non-autonomous homogeneous linear differential equation (27).

Corollary 3.1: Suppose that (27) satisfies Hypothesis 3.1. There exists $K(\tau_0)$ such that

$$\|U(\tau)\| \le K(\tau_0),$$

for all $\tau \in [\tau_0, \infty)$ where $\tau_0 > (\nu/\lambda)^{-1/\nu} > 0$ and $\|\ \|$ represents the corresponding matrix norm.

Corollary 3.2: Suppose that (27) satisfies Hypothesis 3.1. There exists $\hat{K}(\tau_0)$ such that

$$\|U^{-1}(\tau)\| \le \hat{K}(\tau_0),$$

for all $\tau \in [\tau_0, \infty)$ where $\tau_0 > (\nu/\lambda)^{-1/\nu} > 0$ and $\|\ \|$ represents the corresponding matrix norm.

Proposition 3.1 will be seen to be an immediate consequence of the following theorem:

Theorem 3.1: *Consider the system (27) under Hypothesis 3.1. For $\tau_0 > (\nu/\lambda)^{-1/\nu} > 0$ let*

$$\phi(\tau) = \int_{\tau_0}^{\tau} \sqrt{1+F(s)}\,ds.$$

Then, there are complex numbers w_0 and v_0 such that the solution of the system (27) with initial conditions $w(\tau_0) = w_0$ and $v(\tau_0) = v_0$ satisfies

$$w(\tau) = e^{i\phi(\tau)}(1+F(\tau))^{-1/4} + \mathcal{O}(\frac{1}{\tau^{\nu}}),$$

$$v(\tau) = e^{i\phi(\tau)}(1+F(\tau))^{1/4} + \mathcal{O}(\frac{1}{\tau^{\nu}}),$$

for $\tau \ge \tau_0$.

Proof: To prove the theorem we consider the quantities

$$W(\tau) = w(\tau) - e^{i\phi(\tau)}(1+F(\tau))^{-1/4},$$

$$V(\tau) = v(\tau) - e^{i\phi(\tau)}(1+F(\tau))^{1/4},$$

which satisfy differential equations

$$\frac{d}{d\tau}\begin{bmatrix} W \\ iV \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(1+F(\tau)) & 0 \end{bmatrix}\begin{bmatrix} W \\ iV \end{bmatrix} + h(\tau), \tag{29}$$

where

$$h(\tau) = \frac{1}{4}e^{i\phi(\tau)}\frac{F'(\tau)}{(1+F(\tau))^{3/4}}\begin{bmatrix} (1+F(\tau))^{-1} \\ -1 \end{bmatrix}. \tag{30}$$

We observe that the associated homogeneous equation of the inhomogeneous non-autonomous linear differential equation (29)

$$\frac{d}{d\tau}\begin{bmatrix} W \\ iV \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(1+F(\tau)) & 0 \end{bmatrix}\begin{bmatrix} W \\ iV \end{bmatrix} \tag{31}$$

thus according to Corollary 3.1 there exists $K(\tau_0)$, $\tau_0 > (\nu/\lambda)^{-1/\nu} > 0$ such that

$$\|U(\tau)\| \le K(\tau_0),$$

where $\| \ \|$ represents the corresponding matrix norm. Moreover, according to Corollary 3.2 there exists $\hat{K}(\tau_0)$ such that

$$\|U^{-1}(\tau)\| \le \hat{K}(\tau_0),$$

for $\tau_0 > (\nu/\lambda)^{-1/\nu} > 0$. We now use the variation of parameters formula to get the solution of (29)

$$\begin{bmatrix} W(\tau) \\ iV(\tau) \end{bmatrix} = U(\tau)\left[U^{-1}(\tau_0)\begin{bmatrix} W(\tau_0) \\ iV(\tau_0) \end{bmatrix} + \int_{\tau_0}^{\tau} U^{-1}(s)h(s)ds\right].$$

Notice that under the suppositions of Hypothesis 3.1 it follows from (30)

$$h(\tau) \sim \frac{1}{4}e^{i\phi(\tau)}F'(\tau)\left(\begin{bmatrix} 1-\frac{7}{4}F(\tau) \\ -1+\frac{3}{4}F(\tau) \end{bmatrix} + \mathcal{O}\left(F^2(\tau)\right)\right),$$

therefore

$$\|h(\tau)\| \le \frac{\lambda}{\tau^{1+\nu}},$$

for large enough τ. Hence, if we define the initial conditions $W(\tau_0)$, $V(\tau_0)$ by

$$\begin{bmatrix} W(\tau_0) \\ iV(\tau_0) \end{bmatrix} = -U(\tau_0)\int_{\tau_0}^{\infty} U^{-1}(s)h(s)ds,$$

then

$$\begin{bmatrix} W(\tau) \\ iV(\tau) \end{bmatrix} = -\int_{\tau}^{\infty} U(\tau - s)h(s)ds$$

and

$$\left\|\begin{bmatrix} W(\tau) \\ iV(\tau) \end{bmatrix}\right\| = \mathcal{O}\left(\frac{1}{\tau^{\nu}}\right),$$

which completes the proof of the theorem. □

Proposition 3.1 now follows by recasting everything in terms of the original time. We first note that, for large t (and hence for large τ)

$$\phi(\tau) = \int_{\tau_0}^{\tau}\sqrt{1+F(s)}\,ds \sim \tau + \theta,$$

for some real constant θ. Inserting the appropriate constant into (24),

$$A(t) = \sqrt{2}e^{-i\theta}\frac{\sqrt{\alpha(t)}}{q(t)^{1/4}}w(t)$$

and from (22)

$$\begin{aligned} B(t) &= -\frac{i}{2\alpha(t)}\dot{A}(t) \\ &= \frac{q(t)^{1/4}}{\sqrt{2\alpha(t)}}e^{-i\theta}v(t) - ie^{-i\theta}w(t)\frac{2q(t)\dot{\alpha}(t) - \alpha(t)\dot{q}(t)}{4\sqrt{2}q(t)^{5/4}\alpha(t)^{3/2}}. \end{aligned}$$

Now, for large t (and hence for large τ),

$$w(\tau) = e^{i\phi(\tau)}(1+F(\tau))^{-1/4} + \mathcal{O}(\frac{1}{\tau^{\nu}}) \sim e^{i(\tau+\theta)},$$

$$v(\tau) = e^{i\phi(\tau)}(1+F(\tau))^{1/4} + \mathcal{O}(\frac{1}{\tau^{\nu}}) \sim e^{i(\tau+\theta)},$$

so

$$A(t) \sim \frac{\sqrt{2\alpha(t)}}{q(t)^{1/4}}e^{i\int_0^t \sqrt{q(s)}ds}$$

and

$$B(t) \sim \left(\frac{q(t)^{1/4}}{\sqrt{2\alpha(t)}} - \frac{i}{4\sqrt{2}}\frac{2q(t)\dot{\alpha}(t) - \alpha(t)\dot{q}(t)}{q(t)^{5/4}\alpha(t)^{3/2}}\right)e^{i\int_0^t \sqrt{q(s)}ds}$$

and, using (20),

$$|A(t)B(t)| \sim 1$$

as $t \to \infty$. Finally, we note that

$$A(t)\overline{B(t)} + \overline{A(t)}B(t) = 2,$$

since

$$A(t)\overline{B(t)} + \overline{A(t)}B(t) = w(t)\overline{v(t)} + v(t)\overline{w(t)} \sim 2$$

as $t \to \infty$ and the quantity on the left is conserved.

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APPENDIX: BACKGROUND

In this section we give proofs of two lemmas which we used in the text. These results are restatements or at best mild generalizations of those in the classic works of Hille[11] and Courant and Hilbert,[10] and are included here for completeness.

1. Two results of Liouville

We start by recalling two well-known results of Liouville which will be used in the sequel.

Let D be the domain of definition of the (not necessarily linear) differential equation

$$\frac{d\vec{x}}{d\tau} = \vec{v}(\vec{x})$$

in Euclidean space. We denote by $D(\tau)$ the image of the region D under the action of the phase flow and by $Vol(\tau)$ the volume of the region $D(\tau)$.

Theorem A.1 (Liouville 's theorem):

$$\frac{dVol}{d\tau} = \int_{D(\tau)} \text{div } \vec{v} \, ds.$$

Corollary A.1: If div $\vec{v} = 0$, *then the phase flow preserves the volume of any region.*

Proposition A.1: **(Liouville 's formula)** *Suppose that* $\tau \mapsto U(\tau)$ *is a matrix solution of the homogeneous linear system*

$$\frac{d\vec{x}}{d\tau} = \Lambda(\tau)\vec{x}$$

on the open interval $\mathbb{I}$. *If* $\tau_0 \in \mathbb{I}$, *then*

$$\det U(\tau) = \det U(\tau_0) e^{\int_{\tau_0}^{\tau} tr \ \Lambda(s) ds}.$$

2. Asymptotics of Sturm-Liouville equations

In this subsection we study the asymptotic behavior of the solutions of Sturm-Liouville equations

$$w'' + (1 + F(\tau))\, w = 0 \tag{A1}$$

following Courant and Hilbert[10] as well as the consequences to the fundamental matrix solution of the associated non-autonomous homogeneous linear differential equation

$$\frac{d}{d\tau}\begin{bmatrix} w \\ iv \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(1 + F(\tau)) & 0 \end{bmatrix}\begin{bmatrix} w \\ iv \end{bmatrix} \tag{A2}$$

obtained from (A1) by introducing the new variable $w' = iv$. All derivatives in this section are with respect to the time τ.

Hypothesis A.1: Suppose that $F(\tau) \in \mathbf{C}^1$ *such that*

$$|F(\tau)| < \frac{\lambda}{\nu \tau^{\nu}}, \quad |F'(\tau)| < \frac{\lambda}{\tau^{1+\nu}},$$

where λ *and* ν *are positive constants.*

Lemma A.1: Suppose that (A1) satisfies Hypothesis A.1 then every solution of Eq. (A1) is bounded for $\tau \to \infty$.

Proof: Following Courant and Hilbert[10] (page 331), we multiply (A1) by w' and integrate from a positive lower bound τ_0 (to be suitably determined later) to τ. We obtain

$$\int_{\tau_0}^{\tau} (w''w' + ww')\, ds = -\int_{\tau_0}^{\tau} F(\tau) w w' \, ds,$$

$$\left(\frac{w'^2}{2} + \frac{w^2}{2}\right)\Bigg|_{\tau_0}^{\tau} = -\int_{\tau_0}^{\tau} F(\tau) w \, dw,$$

$$w'^2\Big|_{\tau_0}^{\tau} + w^2\Big|_{\tau_0}^{\tau} = -2\int_{\tau_0}^{\tau} F(s) w w' \, ds = -F(s)w^2\Big|_{\tau_0}^{\tau} + \int_{\tau_0}^{\tau} F'(s) w^2 \, ds.$$

From this it follows immediately that

$$w^2(\tau) \le w'^2(\tau) + w^2(\tau) \le \Theta(\tau_0) + |F(\tau)|\, w^2(\tau) + \int_{\tau_0}^{\tau} \left|F'(s)\right| w^2 \, ds, \tag{A3}$$

where

$$\Theta(\tau_0) = w'^2(\tau_0) + w^2(\tau_0) + |F(\tau_0)|\, w^2(\tau_0)$$

denotes an expression which depends only on the lower limit τ_0. Let $M(\tau)$ be the maximum of the function $w(s)$ in the interval $\tau_0 \leq s \leq \tau$, assumed at the point $\bar{s}$ then from (A3) and (28) it follows that

$$w^2(\bar{s}) \leq \Theta(\tau_0) + \frac{w^2(\bar{s})\lambda}{\nu \bar{s}^{\nu}} + w^2(\bar{s}) \int_{\tau_0}^{\bar{s}} \frac{\lambda}{s^{1+\nu}}\, ds,$$

i.e.,

$$M^2(\tau) \leq \Theta(\tau_0) + \frac{M^2(\tau)\lambda}{\nu \bar{s}^{\nu}} + \frac{M^2(\tau)\lambda}{\nu}\left(\frac{1}{\tau_0^{\nu}} - \frac{1}{\bar{s}^{\nu}}\right)$$

and thus that

$$M^2(\tau)\left(1 - \frac{\lambda}{\nu \tau_0^{\nu}}\right) \leq \Theta(\tau_0).$$

Now if we take $\tau_0 > (\nu/\lambda)^{-1/\nu} > 0$, we obtain

$$|M(\tau)| \leq \sqrt{\frac{\Theta(\tau_0)}{\left(1 - \frac{\lambda}{\nu \tau_0^{\nu}}\right)}}$$

as a bound for $|M(\tau)|$ independent of τ. □

Let $U(\tau)$ be *fundamental matrix solution* of the associated non-autonomous homogeneous linear differential equation (A2).

Corollary A.2: Suppose that (A2) satisfies Hypothesis A.1. There exists $K(\tau_0)$ *such that*

$$\|U(\tau)\| \leq K(\tau_0),$$

for all $\tau \in [\tau_0, \infty)$ *where* $\tau_0 > (\nu/\lambda)^{-1/\nu} > 0$ *as in the proof of Lemma A.1 and* $\|\ \|$ *represents the corresponding matrix norm.*

Proof: Note that every solution of (A2) is a solution of (A1) and vice versa. Therefore under the conditions of Hypothesis A.1 every solution of Eq. (A2) is bounded for $\tau \in (\tau_0, \infty)$ where τ_0 is the same as in proof of Lemma A.1 thus Lyapunov stable. Lyapunov stability implies that for every ϵ there is a $\delta = \delta(\epsilon, \tau_0)$ such that

$$\left\| U(\tau)U^{-1}(\tau_0)\begin{bmatrix} w(\tau_0) \\ i v(\tau_0) \end{bmatrix} \right\| < \epsilon,$$

for

$$\left\| \begin{bmatrix} w(\tau_0) \\ i v(\tau_0) \end{bmatrix} \right\| < \delta(\epsilon, \tau_0).$$

Let

$$x(\tau) = \begin{bmatrix} w(\tau) \\ i v(\tau) \end{bmatrix},$$

then

$$\|U(\tau)U^{-1}(\tau_0)\| = \sup_{\|\xi\| \leq 1} \left| U(\tau)U^{-1}(\tau_0)\xi \right| = \sup_{\|x(\tau_0)\| \leq \delta} \left| U(\tau)U^{-1}(\tau_0)x(\tau_0)\delta^{-1} \right| \leq \frac{\epsilon}{\delta},$$

for $\tau \geq \tau_0$ which in turn implies $\|U(\tau)\| \leq K(\tau)$. □

We note that the flow of the linear system (A2) is volume preserving according to Liouville 's theorem since its divergence is given as

$$tr\left(\begin{bmatrix} 0 & 1 \\ -(1+F(\tau)) & 0 \end{bmatrix}\right) = 0.$$

It follows immediately the fundamental matrix solution $U(\tau)$ of system (A2) is invertible for all τ and is given as

$$U^{-1}(\tau) = \frac{1}{\det U(\tau)} \operatorname{adj} U(\tau).$$

Corollary A.3: Suppose that (A2) satisfies Hypothesis A.1. There exists $\hat{K}(\tau_0)$ *such that*

$$\|U^{-1}(\tau)\| \leq \hat{K}(\tau_0),$$

for all $\tau \in [\tau_0, \infty)$ *where* $\tau_0 > (\nu/\lambda)^{-1/\nu} > 0$ *as in the proof of Lemma A.1.*

Proof: From (A2) and Corollary A.2 it follows that it is sufficient to show that $\tau \mapsto \int_{\tau_0}^{\tau} tr\, \Lambda(s)\, ds$ is bounded below, where

$$\Lambda(s) = \left(\begin{bmatrix} 0 & 1 \\ -(1+F(s)) & 0 \end{bmatrix}\right).$$

That is certainly true since $tr\, \Lambda(s) = 0$. □

We now consider a non-homogeneous system

$$\frac{d}{d\tau}\begin{bmatrix} w \\ iv \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(1+F(\tau)) & 0 \end{bmatrix}\begin{bmatrix} w \\ iv \end{bmatrix} + \begin{bmatrix} h_1(\tau) \\ h_2(\tau) \end{bmatrix}, \tag{A4}$$

with the homogeneous part (A2) and prove that under some certain restrictions on the function

$$h(\tau) = \begin{bmatrix} h_1(\tau) \\ h_2(\tau) \end{bmatrix}$$

its solutions remain bounded for $\tau \in [\tau_0, \infty)$.

Lemma A.2: Suppose that the homogeneous part (A2) of the system (A4) satisfies Hypothesis A.1 and that

$$\|h(\tau)\| \leq \frac{C}{\tau^{1+\mu}}, \quad \mu > 0,$$

for some $C > 0$, $\tau \in [\tau_0, \infty)$*, and* $\tau_0 > (\nu/\lambda)^{-1/\nu} > 0$ *as in the proof of Lemma A.1, then every solution of (A4) is bounded as* $\tau \to \infty$ *with a bound depending only on* λ, ν, *and* μ.

Proof: If $U(\tau)$ is the fundamental matrix solution of (A2) then every solution $\hat{x}(\tau)$ of (A4) is given by the *variation of parameters formula*

$$\hat{x}(\tau) = U(\tau)\left[U^{-1}(\tau_0)\hat{x}(\tau_0) + \int_{\tau_0}^{\tau} U^{-1}(s)h(s)ds\right].$$

From (A2), Corollary A.2, and Corollary A.3 it follows

$$\|\hat{x}(\tau)\| \leq K(\tau_0)\left(\|U^{-1}(\tau_0)\hat{x}(\tau_0)\| + \hat{K}(\tau_0)\int_{\tau_0}^{\tau} \frac{1}{s^{1+\mu}} ds\right),$$

which in turn implies

$$\|\hat{x}(\tau)\| \leq K(\tau_0)\left(\|U^{-1}(\tau_0)\hat{x}(\tau_0)\| + \hat{K}(\tau_0)\frac{1}{\mu \tau_0^{\mu}}\right) \tag{A5}$$

as $\tau \to \infty$. Equation (A5) can be rewritten as

$$\|\hat{x}(\tau)\| \leq K(\tau_0)\left(\|U^{-1}(\tau_0)\hat{x}(\tau_0)\| + \hat{K}(\tau_0)\frac{1}{\mu}\left(\frac{\nu}{\lambda}\right)^{\mu/\nu}\right),$$

where we chose to keep writing τ_0 even though in reality it can be expressed as a function of λ and ν. □

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