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Dynamics of uncertainties for bound one-dimensional semiclassical wave packets

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We study the time evolution of the uncertainties Δx and Δp in position and momentum, respectively, associated with the semiclassical propagation of certain Gaussian initial states. We show that these quantities behave generically as $|P_1(t) + tP_2(t)|$, where P_1 and P_2 are periodic in time with period that of an underlying classical trajectory. We also show that, despite the overall (generically quadratic) growth in time, the uncertainty product $\Delta x \Delta p$ achieves its minimum of $\hbar/2$ at arbitrarily large times. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4962926]

I. INTRODUCTION

In this study, we consider certain semiclassical ($\hbar \searrow 0$) asymptotic solutions of the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t), \tag{1}$$

where $x \in \mathbb{R}$, $t > 0$, and $V(x)$ is a twice continuously differentiable real valued function. Our main interest is in the time dependence of the uncertainties Δx and Δp in position and momentum, respectively, associated with the state $\Psi(x,t)$ and among our results is that, for certain Gaussian initial states $\Psi(x,0)$, these quantities behave generically as $|P_1(t) + tP_2(t)|$, where P_1 and P_2 are periodic in time with period that of an underlying classically bound motion. We also show that, despite the overall (generically quadratic) growth in time, the uncertainty product $\Delta x \Delta p$ achieves its minimum of $\hbar/2$ at arbitrarily large times.

We let $\langle\cdot,\cdot\rangle$ and $||\cdot||$ denote the inner product and norm for $L^2(\mathbb{R}, dx)$. Given $\Psi(x,t) \in L^2(\mathbb{R}, dx)$ for $t \geq 0$ normalized so that

$$\langle\Psi,\Psi\rangle = \int_{-\infty}^{\infty} |\Psi(x,t)|^2\, dx = 1$$

the expectations $\langle x\rangle$ and $\langle p\rangle$ of position and momentum are given by

$$\langle x\rangle = \langle\Psi, x\Psi\rangle$$

and

$$\langle p\rangle = \left\langle\Psi, (-i\hbar\frac{\partial}{\partial x})\Psi\right\rangle$$

and the uncertainties in these quantities by

$$\Delta x = \sqrt{\langle\Psi, (x - \langle x\rangle)^2\Psi\rangle}$$

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and

$$\Delta p = \sqrt{\left\langle \Psi, (-i\hbar\frac{\partial}{\partial x} - \langle p \rangle)^2 \Psi \right\rangle}.$$

The uncertainty principle

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

follows, and if equality is achieved we will say that the state Ψ is of minimal uncertainty.

The asymptotic solutions we consider are Hagedorn's[1] "semi-classical" wave packet states $\{\varphi_n\}$.

Definition 1. For $n \in \mathbb{Z}^+ \cup \{0\}$, a, $\eta \in \mathbb{R}$, $\hbar > 0$, and A, $B \in \mathbb{C}$ satisfying

$$A\overline{B} + \overline{A}B = 2, \tag{2}$$

we let

$$\begin{aligned}\varphi_n(A, B, \hbar, a, \eta, x) = {} & 2^{-n/2}(n!)^{-1/2}(\pi\hbar)^{-1/4} A^{-(n+1)/2}\overline{A}^{n/2} \\ & \times H_n(\hbar^{-1/2}|A|^{-1}(x-a)) \\ & \times \exp\left\{-\frac{1}{2\hbar}BA^{-1}(x-a)^2 + \frac{i}{\hbar}\eta(x-a)\right\},\end{aligned}$$

where H_n is the n th order Hermite polynomial.

The condition (2) is not truly restrictive, for any complex number Γ with $\mathrm{Re}(\Gamma) > 0$ can be written in the form $\Gamma = BA^{-1}$ for some A and B satisfying (2). The purpose of writing $\Gamma = BA^{-1}$ with A and B satisfying (2) is to ensure that

$$\mathrm{Re}(BA^{-1}) = |A|^{-2} \text{ and } \mathrm{Re}(AB^{-1}) = |B|^{-2}$$

which in turn implies that the state $\varphi_n(A, B, \hbar, a, \eta, x)$ is normalized in $L^2(\mathbb{R}, dx)$ and has expected position and momentum

$$\langle x \rangle = a \text{ and } \langle p \rangle = \eta$$

and uncertainties

$$\Delta x = \sqrt{\hbar\left(n + \frac{1}{2}\right)}\,|A| \text{ and } \Delta p = \sqrt{\hbar\left(n + \frac{1}{2}\right)}\,|B|\,.$$

It follows easily from condition (2) that $|A|\,|B| \geq 1$.

Now, suppose that V is twice continuously differentiable, bounded from below, and such that $|V(x)| \leq Ce^{Mx^2}$ for some constants C and M, so that $H = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + V(x)$ generates a unitary propagator $U(t) = e^{-itH/\hbar}$ on $L^2(\mathbb{R}, dx)$. Let

$$\begin{aligned}\varphi(\hbar, x, t) &= e^{iS(t)/\hbar}\varphi_0(A(t), B(t), \hbar, a(t), \eta(t), x) \\ &= (\pi\hbar)^{-1/4} A(t)^{-1/2} \exp\left\{-\frac{1}{2\hbar}B(t)A(t)^{-1}(x-a(t))^2 \right. \\ &\qquad \left. + \frac{i}{\hbar}\eta(t)(x-a(t)) + \frac{i}{\hbar}S(t)\right\},\end{aligned} \tag{3}$$

where $\{a(t), \eta(t), A(t), B(t), S(t)\}$ is the solution of

$$\dot{a}(t) = \eta(t), \tag{4}$$

$$\dot{\eta}(t) = -V'(a(t)), \tag{5}$$

$$\dot{A}(t) = iB(t), \tag{6}$$

$$i\dot{B}(t) = -V''(a(t))A(t), \tag{7}$$

$$\dot{S}(t) = \frac{1}{2}\eta(t)^2 - V(a(t)), \tag{8}$$

for some initial condition $\{a(0), \eta(0), A(0), B(0), S(0)\} \in \mathbb{R}\times\mathbb{R}\times\mathbb{C}\times\mathbb{C}\times\{0\}$ with $A(0)\overline{B(0)} + \overline{A(0)}B(0) = 2$. Let the branch of the square root of $A(t)$ be determined by continuity in t starting on the principle branch at $t = 0$. Then[1]

$$\left\| e^{-itH/\hbar}\varphi(\hbar,\cdot,0) - \varphi(\hbar,\cdot,t)\right\| = O(\hbar^{1/2})$$

for t in any compact interval $[0,T]$. Note that (2) is conserved, that is,

$$A(t)\overline{B(t)} + \overline{A(t)}B(t) = 2$$

for $t \geq 0$ and that this particular asymptotic solution $\varphi(\hbar, x, t)$ has expected position and momentum

$$\langle x\rangle = a(t) \text{ and } \langle p\rangle = \eta(t)$$

and uncertainties

$$\Delta x = \sqrt{\frac{\hbar}{2}}\,|A(t)| \text{ and } \Delta p = \sqrt{\frac{\hbar}{2}}\,|B(t)|\,.$$

For the remainder of this paper, we study the evolution of the quantities $A(t)$ and $B(t)$.

II. CLASSICAL MECHANICS AND THE EVOLUTION OF UNCERTAINTIES

Let $(a(a_0,\eta_0,t), \eta(a_0,\eta_0,t))$ be the solution of the system (4)-(5) subject to initial conditions a_0, $\eta_0 \in \mathbb{R}$. We denote by E the (conserved) energy $E = \frac{1}{2}\eta_0^2 + V(a_0)$ and consider initial conditions (a_0, η_0) such that a_0 is between two adjacent roots $x_- < x_+$ of $V(x) = E$ with $V'(x_-) < 0$ and $V'(x_+) > 0$. Under such conditions $a(t)$ and $\eta(t)$ are periodic with a positive minimal period τ depending smoothly on the energy,[2]

$$\tau(E) = \sqrt{2}\int_{x_-(E)}^{x_+(E)} \frac{1}{\sqrt{E - V(x)}}\,dx.$$

Note in particular that by assumption not both η_0 and $V'(a_0)$ are zero.

Under all the above hypotheses we want to study the system (6)-(7) with complex initial conditions A_0 and B_0 satisfying

$$\overline{A_0}B_0 + A_0\overline{B_0} = 2.$$

In other words, we consider the Hill's equation

$$\ddot{A}(t) + V''(a(t))A(t) = 0 \tag{9}$$

(and $B(t) = -i\dot{A}(t)$) or the system

$$\frac{d}{dt}\begin{bmatrix} A(t)\\ iB(t)\end{bmatrix} = \begin{bmatrix} 0 & 1\\ -V''(a(t)) & 0\end{bmatrix}\begin{bmatrix} A(t)\\ iB(t)\end{bmatrix}. \tag{10}$$

A. The Floquet theory

The arguments below are based on the proof of Floquet's theorem[3,4] and some known properties[5] of the system (10). Under the hypotheses above, the system (10) is the equation of variation[6] for the system (4)-(5) and its fundamental matrix solution $\Phi(t)$ satisfies

$$\frac{d}{dt}\Phi(t) = \begin{bmatrix} 0 & 1\\ -V''(a(t)) & 0\end{bmatrix}\Phi(t),\ \Phi(0) = \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}$$

and is given by

$$\Phi(t) = \begin{bmatrix} \frac{\partial a}{\partial a_0}(t) & \frac{\partial a}{\partial \eta_0}(t)\\ \frac{\partial \eta}{\partial a_0}(t) & \frac{\partial \eta}{\partial \eta_0}(t)\end{bmatrix},$$

so

$$A(t) = \frac{\partial a}{\partial a_0} A_0 + i \frac{\partial a}{\partial \eta_0} B_0,$$
$$B(t) = \frac{\partial \eta}{\partial \eta_0} B_0 - i \frac{\partial \eta}{\partial a_0} A_0.$$

Differentiating

$$a(a_0, \eta_0, t) = a(a_0, \eta_0, t + \tau)$$

with respect to a_0 gives

$$\frac{\partial a}{\partial a_0}(t) = \frac{\partial a}{\partial a_0}(t + \tau) + \eta(t + \tau) \frac{\partial \tau}{\partial a_0}.$$

Using $\frac{\partial \tau}{\partial a_0} = \frac{d\tau}{dE} V'(a_0)$ and setting $t = 0$,

$$1 = \frac{\partial a}{\partial a_0}(\tau) + \eta_0 \tau'(E) V'(a_0)$$

and hence

$$\frac{\partial a}{\partial a_0}(\tau) = 1 - \eta_0 \tau'(E) V'(a_0).$$

Through similar calculations we obtain the monodromy matrix

$$\Phi(\tau) = \begin{bmatrix} 1 - \eta_0 \tau'(E) V'(a_0) & -\eta_0^2 \tau'(E) \\ V'(a_0)^2 \tau'(E) & 1 + \eta_0 \tau'(E) V'(a_0) \end{bmatrix} \tag{11}$$

so that

$$\Phi(t + \tau) = \Phi(t) \Phi(\tau).$$

All solutions corresponding to energy E will be τ-periodic if and only if the monodromy matrix is equal to the identity matrix, and this occurs if and only if $\tau'(E) = 0$. Though this case is rather non-generic, we remark that there are potentials for which $\tau'(E) = 0$ for all allowable E, for example, the harmonic oscillator or more generally the class of isochronous potentials.[7]

Note that $\Phi(\tau)$ has trace 2 and determinant 1, so the eigenvalues ρ of the monodromy matrix satisfy

$$(1 - \rho)^2 = 0,$$

i.e., $\rho_1 = \rho_2 = 1$. Since at least one of the η_0 and $V'(a_0)$ is nonzero, there is only one eigenvector if $\tau'(E) \neq 0$, namely,

$$\begin{bmatrix} \eta_0 \\ -V'(a_0) \end{bmatrix}$$

and therefore a τ-periodic solution

$$P_1(t) = \begin{bmatrix} \eta(t) \\ -V'(a(t)) \end{bmatrix}$$

and a linearly independent solution $Y(t) = P_2(t) + t P_1(t)$ with P_2 τ-periodic.[8] Hence if $\tau'(E) \neq 0$, then $A(t)$ and $B(t)$ have the form

$$A(t) = A_1(t) + t A_2(t),$$
$$B(t) = B_1(t) + t B_2(t),$$

with $A_1, A_2, B_1,$ and B_2 τ-periodic.

We collect our results so far in the following proposition.

Proposition 2.1. Suppose $V \in C^2(\mathbb{R},\mathbb{R})$. Let $\{a(t),\eta(t),A(t),B(t)\}$ be the solution of (4)-(7) for some initial condition $\{a_0,\eta_0,A_0,B_0\} \in \mathbb{R}\times\mathbb{R}\times\mathbb{C}\times\mathbb{C}$ with $A_0\overline{B_0}+\overline{A_0}B_0 = 2$ and such that a_0 is between two adjacent roots $x_- < x_+$ of $V(x) = E$ with $V'(x_-) < 0$ and $V'(x_+) > 0$. Let $\tau(E)$ denote the period of $a(t)$ and $\eta(t)$. Then, if $\tau'(E) = 0$, $A(t)$ and $B(t)$ are periodic with period τ while if $\tau'(E) \neq 0$, $A(t)$ and $B(t)$ have the form

$$A(t) = A_1(t) + tA_2(t),$$
$$B(t) = B_1(t) + tB_2(t),$$

with A_1, A_2, B_1, and B_2 periodic with period τ.

For notational purposes it is convenient to define a conserved quantity θ as[9]

$$\theta = \theta(t) = A(t)V'(a(t)) + iB(t)\eta(t) = A_0V'(a_0) + iB_0\eta_0.$$

We note that $\theta \neq 0$, for

$$A_0V'(a_0) + iB_0\eta_0 = 0 \Rightarrow iV'(a_0) = \eta_0 B_0 A_0^{-1}$$

and, since $A_0\overline{B_0}+\overline{A_0}B_0 = 2$ implies $\mathrm{Re}(B_0A_0^{-1}) = |A_0|^{-2} > 0$, the equality can hold if and only if $\eta_0 = V'(a_0) = 0$, contradicting our assumptions on the initial conditions (a_0,η_0).

Powers of the monodromy matrix are quite simple to compute: if n is a positive integer then

$$(\Phi(\tau))^n = \begin{bmatrix} 1 - n\eta_0\tau'(E)V'(a_0) & -n\eta_0^2\tau'(E) \\ nV'(a_0)^2\tau'(E) & 1 + n\eta_0\tau'(E)V'(a_0) \end{bmatrix}$$

which in turn implies that at integer multiples of the period we have

$$A(n\tau) = A_0 - n\theta\tau'(E)\eta_0, \tag{12}$$
$$B(n\tau) = B_0 - in\theta\tau'(E)V'(a_0) \tag{13}$$

and, since θ and at least one of the η_0 and $V'(a_0)$ are nonzero, we see that the uncertainty product $|A(n\tau)B(n\tau)| = |A_0B_0| + O(n)$ as $n \to \infty$ unless $\tau'(E) = 0$. If the classical motion starts at a turning point ($\eta_0 = 0$), the position uncertainty of the Gaussian coherent state (3) is recurrent at times $t = n\tau$ and the momentum uncertainty behaves as $O(n|\tau'(E)|)$ as $n \to \infty$. A similar result holds at the bottom of a potential well ($V'(a_0) = 0$, $\eta_0 \neq 0$) with the roles of position and momentum uncertainty interchanged. It also clearly follows that in the general ($\tau'(E) \neq 0$) case the uncertainty product $|A(t)B(t)|$ cannot remain (or be asymptotically) minimal, for at times $n\tau$ this quantity is at least on the order of n and typically on the order of n^2.

B. A polar representation and times of minimal uncertainty

We also have at our disposal a polar representation for $A(t)$, namely, let[4]

$$r(t) = |A(t)|$$

and

$$\omega(t) = \omega_0 + \int_0^t \frac{1}{r(s)^2}\,ds$$

with

$$\tan\omega_0 = \frac{\mathrm{Im}(A_0)}{\mathrm{Re}(A_0)}$$

then

$$A(t) = r(t)e^{i\omega(t)}$$

and

$$B(t) = -i\dot{A}(t) = \frac{1}{r(t)}\left(1 - ir(t)\dot{r}(t)\right)e^{i\omega(t)}.$$

As a quick aside we note that this polar representation allows us to track the branch of the square root of $A(t)$ needed in the specification of Gaussian coherent states of the form (3). The choice of branch is related to the Maslov index[10] of the classical path.

Proposition 2.2. Under the hypothesis of Proposition 2.1, the trajectory $\{A(t) : 0 < t \le \tau\}$ *crosses the negative real axis exactly once. Hence, if we determine the branch of the square root along this trajectory by continuity and use* $\sqrt{\cdot}$ *to denote the principal branch, we have*

$$[A(0)]^{1/2} = \sqrt{A(0)} \Longrightarrow [A(\tau)]^{1/2} = e^{i\pi}\sqrt{A(\tau)}.$$

Proof. Let

$$Y_1(t) = \mathrm{Re}(A(t)) = r(t)\cos(\omega(t)),$$

$$Y_2(t) = \mathrm{Im}(A(t)) = r(t)\sin(\omega(t)).$$

Then,

$$Y_1(t)Y_2'(t) - Y_1'(t)Y_2(t) = 1,$$

so $\{Y_1(t), Y_2(t)\}$ is a fundamental set for the Hill's equation (9). Therefore, $\eta(t)$ is a linear combination of $Y_1(t)$ and $Y_2(t)$, which by explicit calculation is

$$\eta(t) = -|\theta|\; r(t)\sin(\omega(t) - \delta)$$

with $\tan\delta = \frac{\mathrm{Im}(\theta)}{\mathrm{Re}(\theta)}$. The proposition then follows easily using the fact that η is periodic and the increasing function ω increases by π between turning points ($\eta = 0$). ■

We now turn our attention to the perhaps surprising result that, despite the overall (generically quadratic) growth in time, the uncertainty product $|A(t)B(t)|$ achieves its minimum at arbitrarily large times.

Proposition 2.3. Under the hypothesis of Proposition 2.1, given any $T > 0$ *there is a time* $t > T$ *such that* $|A(t)B(t)| = 1$.

Proof. From the condition

$$\overline{A}B + A\overline{B} = 2$$

or

$$\mathrm{Re}(\overline{A}B) = 1,$$

we have

$$|A|\,|B| = |AB| = \left|\overline{A}B\right| = \sqrt{1 + \mathrm{Im}(\overline{A}B)^2}$$

and we see that the uncertainty product $|A|\,|B|$ is minimal if and only if $\mathrm{Im}(\overline{A}B) = 0$. Since $r(t) > 0$ and

$$\frac{d}{dt}r(t)^2 = 2r(t)\dot{r}(t) = -2\mathrm{Im}(B(t)\overline{A(t)}),$$

we have $|A|\,|B| = 1$ if and only if $\dot{r}(t) = 0$. In the periodic ($\tau'(E) = 0$) case, it follows easily from the mean value theorem that $\dot{r}(t) = 0$ at least once per period. We will establish a similar result in the non-periodic ($\tau'(E) \neq 0$) case using the intermediate value theorem by showing that $\dot{r}(t)$ changes sign on a certain interval. Throughout the remainder of the proof, we assume $\tau'(E) \neq 0$. By explicit calculation Equations (12) and (13) imply

$$\begin{aligned} r(n\tau)\dot{r}(n\tau) = r(0)\dot{r}(0) &+ n\tau'(E)\left(r(0)^2V'(a_0)^2 - \eta_0^2\left(\frac{1}{r(0)^2} + \dot{r}(0)^2\right)\right) \\ &- 2n^2\tau'(E)^2 r(0)\dot{r}(0)\eta_0^2 V'(a_0)^2 \\ &- n^2\tau'(E)^2\eta_0 V'(a_0)\left(\eta_0^2\left(\frac{1}{r(0)^2} + \dot{r}(0)^2\right) + V'(a_0)^2 r(0)^2\right). \end{aligned} \tag{14}$$

It is instructive to first consider two special cases.

Case 1: The motion begins at a turning point, that is, $(a_0,\eta_0) = (x_\pm, 0)$, where we use $x_\pm$ to denote either of the turning points x_- or x_+. In this case (14) is

$$r(n\tau)\dot{r}(n\tau) = r(0)\dot{r}(0) + n\tau'(E)r(0)^2V'(x_\pm)^2.$$

Suppose $\dot{r}(0) \neq 0$ and note that if $\dot{r}(0)$ and $\tau'(E)$ have opposite signs, then $\dot{r}(t)$ changes sign on $(0, n\tau)$ for some large enough n, hence, if $\dot{r}(0)\tau'(E) < 0$ then $\dot{r}(t) = 0$ somewhere on $(0, n\tau)$.

Case 2: The motion begins at a critical value of $V(x)$, that is, (a_0,η_0) is such that $V'(a_0) = 0$. In this case (14) is

$$r(n\tau)\dot{r}(n\tau) = r(0)\dot{r}(0) - n\tau'(E)\eta_0^2\left(\frac{1}{r(0)^2} + \dot{r}(0)^2\right).$$

Again suppose $\dot{r}(0) \neq 0$ and note that if $\dot{r}(0)$ and $\tau'(E)$ have the same sign, then $\dot{r}(t)$ changes sign on $(0, n\tau)$ for some large enough n, hence, if $\dot{r}(0)\tau'(E) > 0$ then $\dot{r}(t) = 0$ somewhere on $(0, n\tau)$.

Now we consider the general case. Let t_p and t_c be any two positive times at which the trajectory is at a turning point and at a critical point of the potential, respectively, that is, at t_p we have

$$(a(a_0,\eta_0,t_p), \eta(a_0,\eta_0,t_p)) = (x_\pm, 0)$$

while at t_c we have

$$(a(a_0,\eta_0,t_c), \eta(a_0,\eta_0,t_c)) = (a_c, \eta_c)$$

with $\eta_c = \eta(t_c)$ and $V'(a_c) = 0$. By existence and uniqueness we have

$$\begin{bmatrix} A(t+t_p) \\ iB(t+t_p) \end{bmatrix} = \Phi(t+t_p)\begin{bmatrix} A_0 \\ iB_0 \end{bmatrix} = \Phi_p(t)\begin{bmatrix} A(t_p) \\ iB(t_p) \end{bmatrix}$$

and

$$\begin{bmatrix} A(t+t_c) \\ iB(t+t_c) \end{bmatrix} = \Phi(t+t_c)\begin{bmatrix} A_0 \\ iB_0 \end{bmatrix} = \Phi_c(t)\begin{bmatrix} A(t_c) \\ iB(t_c) \end{bmatrix},$$

where

$$\Phi_p(t) = \begin{bmatrix} \dfrac{\partial a}{\partial a_0}(x_\pm,0,t) & \dfrac{\partial a}{\partial \eta_0}(x_\pm,0,t) \\ \dfrac{\partial \eta}{\partial a_0}(x_\pm,0,t) & \dfrac{\partial \eta}{\partial \eta_0}(x_\pm,0,t) \end{bmatrix}$$

and

$$\Phi_c(t) = \begin{bmatrix} \dfrac{\partial a}{\partial a_0}(a_c,\eta_c,t) & \dfrac{\partial a}{\partial \eta_0}(a_c,\eta_c,t) \\ \dfrac{\partial \eta}{\partial a_0}(a_c,\eta_c,t) & \dfrac{\partial \eta}{\partial \eta_0}(a_c,\eta_c,t) \end{bmatrix}.$$

We now consider the monodromy matrices associated with the fundamental matrices $\Phi_p(t)$ and $\Phi_c(t)$. The calculations leading to (11) give

$$\Phi_p(\tau) = \begin{bmatrix} 1 & 0 \\ V'(x_\pm)^2\tau'(E) & 1 \end{bmatrix}$$

and

$$\Phi_c(\tau) = \begin{bmatrix} 1 & -\eta(t_c)^2\tau'(E) \\ 0 & 1 \end{bmatrix}$$

giving us

$$\begin{aligned} A(t_p + n\tau) &= A(t_p), \\ B(t_p + n\tau) &= B(t_p) - in\theta(t_p)\tau'(E)V'(x_\pm) \end{aligned}$$

and

$$A(t_c + n\tau) = A(t_c) - n\theta(t_c)\tau'(E)\eta(t_c),$$
$$B(t_c + n\tau) = B(t_c)$$

which in turn lead to

$$r(t_p + n\tau)\dot{r}(t_p + n\tau) = r(t_p)\dot{r}(t_p) + n\tau'(E)r(t_p)^2 V'(x_\pm)^2 \tag{15}$$

and

$$r(t_c + n\tau)\dot{r}(t_c + n\tau) = r(t_c)\dot{r}(t_c) - n\tau'(E)\eta(t_c)^2\left(\frac{1}{r(t_c)^2} + \dot{r}(t_c)^2\right). \tag{16}$$

Now, if $\dot{r}(t_p) = 0$ or $\dot{r}(t_c) = 0$ or if $\dot{r}(t_p)$ and $\dot{r}(t_c)$ have opposite signs then we are done, otherwise if both $\dot{r}(t_p)$ and $\dot{r}(t_c)$ have the opposite sign of $\tau'(E)$ then (15) leads to a change of sign in $\dot{r}(t)$ for large enough n while if they both have the same sign as $\tau'(E)$, then (16) leads to a change of sign in $\dot{r}(t)$ for large enough n. Due to the arbitrariness of t_p and t_c, it follows that given any $T > 0$ there is a time $t > T$ for which $\dot{r}(t) = 0$. ■

Consider for the moment a single-well potential (such as the Morse potential in Sec. III) with $\tau'(E) > 0$. Let t_{p1} be the first time at which the trajectory is at $(x_-, 0)$. Let t_{c1} be the first time after t_{p1} for which the trajectory crosses the minimum of the potential, t_{p2} be the first time after t_{c1} for which the trajectory is at $(x_+, 0)$, and t_{c2} the first time after t_{p2} for which the trajectory again crosses the minimum of the potential. From ((15) and (16)) there exists N such that $\dot{r}(t_{p1,2} + n\tau) > 0$ and $\dot{r}(t_{c1,2} + n\tau) < 0$ for $n \geq N$, hence if t is large enough we have $|A(t)B(t)| = 1$ at least four times per period, *i.e.,* once for each piece of the trajectory lying between a turning point and a minimum of the potential.

Since the argument in the paragraph above only relies on the existence of a critical value of the potential, we have the following proposition.

Proposition 2.4. Under the hypothesis of Proposition 2.1, if $\tau'(E) \neq 0$ *then there exists* $T > 0$ *such that, for any* $t_0 > T$*, the uncertainty product is minimal (*$|A(t)B(t)| = 1$*) for at least four times* t *in the interval* $[t_0, t_0 + \tau]$.

III. AN EXAMPLE: THE MORSE POTENTIAL

The Morse potential

$$V(x) = D(1 - e^{-\alpha(x - x_0)})^2 - D$$

provides us with an explicitly solvable example of a nonlinear oscillator.[11,12] Consider the system

$$\dot{a}(t) = \eta(t),$$
$$\dot{\eta}(t) = -V'(a(t)) = -2\alpha D e^{-\alpha(a(t) - x_0)}(e^{-\alpha(a(t) - x_0)} - 1)$$

with initial conditions $a_0, \eta_0 \in \mathbb{R}$ such that

$$-D < E(a_0, \eta_0) = \frac{1}{2}\eta_0^2 + V(a_0) < 0.$$

If

$$\kappa = \sqrt{2\,|E|}\alpha$$

and

$$\delta = -\frac{\eta_0}{\alpha\sqrt{2\eta_0^2\,|E|}} \arccos\left(\sqrt{\frac{D}{D + E}}\left[1 + \frac{E}{D}e^{\alpha(a_0 - x_0)}\right]\right),$$

then

$$a(t) = x_0 + \frac{1}{\alpha}\log\left(\frac{D}{|E|}\left[1 - \sqrt{\frac{D+E}{D}}\cos\left(\kappa(t-\delta)\right)\right]\right)$$

satisfies

$$\ddot{a}(t) = -V'(a(t)),$$
$$a(0) = a_0,$$
$$\dot{a}(0) = \eta_0 \neq 0.$$

Note that $a(t)$ is periodic with period

$$\tau(E) = \frac{\pi}{\alpha}\sqrt{\frac{2}{|E|}}.$$

From the explicit representation of a, we can explicitly construct the remaining quantities of interest,

$$\eta(t) = \dot{a}(t),$$
$$A(t) = \frac{\partial a}{\partial a_0}A_0 + i\frac{\partial a}{\partial \eta_0}B_0,$$
$$B(t) = -i\dot{A}(t).$$

For example, for $\eta_0 \neq 0$, explicit calculation shows

$$\frac{\partial a}{\partial a_0} = \frac{\alpha_1 + \alpha_2\cos\left(\kappa(t-\delta)\right) + \alpha_3\sin\left(\kappa(t-\delta)\right) + \alpha_4 t\sin\left(\kappa(t-\delta)\right)}{\alpha D\left(1 - \sqrt{\frac{D+E}{D}}\cos\left(\kappa(t-\delta)\right)\right)},$$

where

$$\alpha_1 = -\frac{D}{E}\frac{\partial E}{\partial a_0},$$
$$\alpha_2 = \frac{2D+E}{2E}\sqrt{\frac{D}{D+E}}\frac{\partial E}{\partial a_0},$$
$$\alpha_3 = D\sqrt{\frac{D+E}{D}}\left(\delta\frac{\partial \kappa}{\partial a_0} - \kappa\frac{\partial \delta}{\partial a_0}\right),$$
$$\alpha_4 = D\sqrt{\frac{D+E}{D}}\frac{\partial \kappa}{\partial a_0},$$

and

$$\frac{\partial a}{\partial \eta_0} = \frac{\beta_1 + \beta_2\cos\left(\kappa(t-\delta)\right) + \beta_3\sin\left(\kappa(t-\delta)\right) + \beta_4 t\sin\left(\kappa(t-\delta)\right)}{\alpha D\left(1 - \sqrt{\frac{D+E}{D}}\cos\left(\kappa(t-\delta)\right)\right)},$$

where

$$\beta_1 = -\frac{D}{E}\frac{\partial E}{\partial \eta_0},$$
$$\beta_2 = \frac{2D+E}{2E}\sqrt{\frac{D}{D+E}}\frac{\partial E}{\partial \eta_0},$$
$$\beta_3 = -D\sqrt{\frac{D+E}{D}}\left(\delta\frac{\partial \kappa}{\partial \eta_0} + \kappa\frac{\partial \delta}{\partial \eta_0}\right),$$
$$\beta_4 = D\sqrt{\frac{D+E}{D}}\frac{\partial \kappa}{\partial \eta_0}.$$

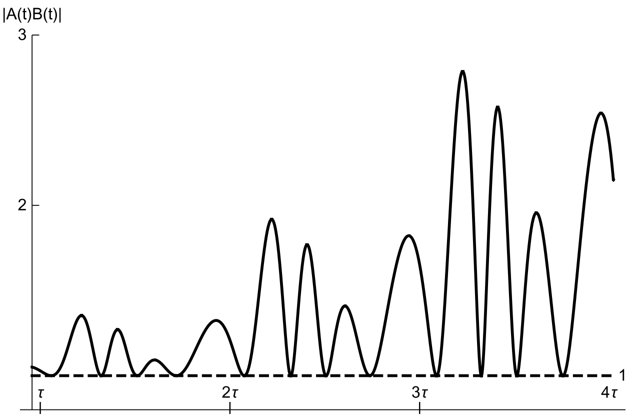

FIG. 1. The uncertainty product $|A(t)B(t)|$ for the Morse potential with $x_0 = 0$, $D = \alpha = 1$, and initial conditions $a_0 = 1/4$, $\eta_0 = -1/4$, and $A_0 = B_0 = 1$.

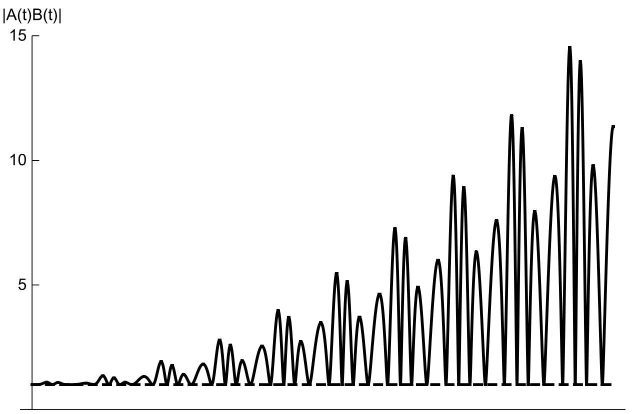

FIG. 2. The uncertainty product $|A(t)B(t)|$ for the Morse potential with the same parameters and initial conditions as in Figure 1 plotted over the interval $[0, 10\tau]$.

Figure 1 shows a plot of $|A(t)B(t)|$ on the interval $\tau \leq t \leq 4\tau$ where we have assigned the parameters in V the values $x_0 = 0$, $D = \alpha = 1$, and used the initial conditions $a_0 = 1/4, \eta_0 = -1/4$, and $A_0 = B_0 = 1$. Evident in this figure is the existence of four times per period where the uncertainty product is minimal. Figure 2 shows a plot with the same parameters and initial conditions but over the interval $0 \leq t \leq 10\tau$ illustrating the overall quadratic growth in time.

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