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REGULARIZATION OF SIMULTANEOUS BINARY COLLISIONS IN SOME GRAVITATIONAL SYSTEMS

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ABSTRACT. In this paper we construct coordinate transforms that regularize the singularities of simultaneous binary collisions in a pair of decoupled Kepler problems and in a restricted collinear four-body problem. This is the first time regularization transforms are introduced for collisions involving more than one colliding pair in the study of the Newtonian gravitational systems.

1. Introduction. Let us start with the equations of the collinear four-body problem. We assume that the physical space is one-dimensional, and x_1, x_2, x_3 and x_4 are the respective positions of four gravitational masses m_1, m_2, m_3 and m_4. Let the interactions be governed by the Newtonian law of gravitations. Then we obtain the following set of ordinary differential equations: let $k = 1, 2, 3$ and 4,

$$m_k \frac{d^2 x_k}{dt^2} = \frac{\partial U}{\partial x_k} \tag{1}$$

where U is the potential function,

$$U = \sum_{1 \leq j < i \leq 4} \frac{m_i m_j}{|x_i - x_j|}.$$

We call the space of $x = (x_1, x_2, x_3, x_4) \in \mathbf{R}^4$ the space of positions. Let $\Delta_{ij} := \{x \in \mathbf{R}^4,\ x_i = x_j\}$ and $\Delta := \cup_{1 \leq j < i \leq 4} \Delta_{ij}$. The potential function U, and consequently equation (1) are singular on Δ.

Let $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be a solution of equation (1) defined on (t_1, t_2), and assume that $x(t) \to L = (L_1, L_2, L_3, L_4)$ as $t \to t_2^-$. We say that $x(t)$ has a singularity of collision at $t = t_2$ if $L \in \Delta$. According to the locations of L in Δ, singularities of collision

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are put into the categories of (a) binary collisions, (b) simultaneous binary collisions, (c) triple collisions and (d) four-body (total) collision. Categories (a)–(d) are in fact the only singularities allowed by equation (1) for $x(t)$. In particular, we have a singularity of *simultaneous binary collision* if L is such that $L_1 = L_2$, $L_3 = L_4$ but $L_1 \neq L_3$. Let us denote the set of L satisfying these restrictions as $\Delta_{12,34}$.

If a solution $x(t)$ has a singularity of binary collision at t_2, then there exists a new time s and an analytic function $t = t(s)$, such that for some $s_2 < \infty$ satisfying $t(s_2) = t_2$, $x(t(s))$ as a function of s is analytic at s_2. In fact, it is well known that the singularities of binary collision in (1) are easily removed by a change of variables, a process commonly referred to as *regularization of collisions*.

The regularization of binary collisions played a pivotal role in Sundman's construction of global power series solutions for the three-body problem. Partly through the influences of Sundman's work ([**7, 9**]), regularization became an important theme. It turned out that the singularity of collisions of three bodies or more is entirely different from that of two bodies. They are in general not regularizable. This was originally proved by Siegel [**6**]. The underlining implications of Siegel's analysis on the phase space geometry have been thoroughly investigated through the introduction of McGehee's transformation [**4**], made possible much progress, including the proofs on the existence of non-collision singularities [**3, 11**], and the construction of global power series solutions [**10**].

As to the issue of regularization, the singularity of simultaneous binary collisions is the only case left open for investigations. On one hand, studies based on Siegel's analysis and McGehee's transformation [**1, 2, 5, 8**] have confirmed that the phase space geometry surrounding the solutions of simultaneous binary collisions are almost identical to that of two independent binary collisions. On the other hand, no regularization transforms have been constructed so far, not even for the system of decoupled Kepler problems.

In this paper we construct coordinate transforms that remove the singularities of simultaneous binary collisions in a pair of decoupled Kepler problems and in a restricted collinear four-body problem. To the best of our knowledge, this is the first time regularization transforms are introduced for collisions involving more than one colliding pair in the study of Newtonian gravitational systems. On the other hand, because of a hurdle posted by interactions between different colliding pairs, we are not yet able to extend our construction of regularization variables to the collinear four-body problem.

Let us now turn to two gravitational systems in which the regularization transforms are constructed in this paper. Decoupled Kepler problems, obtained by dropping the interactions between mass groups $\{m_1, m_2\}$ and $\{m_3, m_4\}$ in the collinear four-body problem, is studied in Section 2. The restricted collinear four-body problem, obtained by letting $m_1 = m_4 = 0$, is studied in subsections 3.1–3.3. For a precise statement of results, see Theorem 1 in subsection 2C and Theorem 2 in subsection 3.3. The difficulty we mentioned earlier in extending the same construction to the collinear four-body problem is discussed in subsection 3.4.

2. On a pair of decoupled Kepler problems. Let $x_1, x_2 \in \mathbf{R}^+$. In this section we study the following set of differential equations

$$\frac{d^2x_1}{dt^2} = -\frac{1}{x_1^2}, \qquad \frac{d^2x_2}{dt^2} = -\frac{1}{x_2^2}. \tag{2}$$

These are the equations for a pair of decoupled Kepler problems. Let

$$v_1 = \frac{dx_1}{dt}, \qquad v_2 = \frac{dx_2}{dt}.$$

We take (x_1, x_2, v_1, v_2) as phase variables to rewrite equation (2) as

$$\frac{dx_1}{dt} = v_1, \qquad \frac{dv_1}{dt} = -\frac{1}{x_1^2}; \qquad \frac{dx_2}{dt} = v_2, \qquad \frac{dv_2}{dt} = -\frac{1}{x_2^2}. \tag{3}$$

For $i = 1, 2$, let $\Delta_i = \{(x_1, x_2) \in \overline{(\mathbf{R}^2)^+}, x_i = 0\}$, $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta_{1,2} = \Delta_1 \cap \Delta_2$. Positions in $\Delta \setminus \Delta_{1,2}$ are positions of *binary collision*, and those in $\Delta_{1,2}$ are positions of *simultaneous binary collisions.*

A. Preliminaries. Let us denote $\mathbf{p} = (x_1, x_2, v_1, v_2)$. Equation (3) has two first integrals of energy

$$\alpha_1 = \frac{v_1^2}{2} - \frac{1}{x_1}, \qquad \alpha_2 = \frac{v_2^2}{2} - \frac{1}{x_2}. \tag{4}$$

Let

$$\mathcal{U}_\rho := \{\mathbf{p} = (x_1, x_2, v_1, v_2) \in (\mathbf{R}^2)^+ \times \mathbf{R}^2 : \ |\alpha_1 x_1|, \ |\alpha_2 x_2| < \rho\},$$

where $\rho < 1$ is positive. Throughout this section we fix ρ and consider only solutions of equation (3) in $\mathcal{U}_\rho$. We also let

$$F(\alpha, u) = \int_0^u \frac{du}{\sqrt{2(\alpha + (1/u))}} = \frac{\sqrt{2}}{3} u^{3/2} \left[1 + X(\alpha, u)\right],$$

where

$$X(\alpha, u) = \sum_{n=1}^{\infty} \frac{3c_n}{2n+3} \alpha^n u^n \tag{5}$$

and c_n are such that

$$(1+x)^{-1/2} = \sum_{n=0}^{\infty} c_n x^n. \tag{6}$$

Lemma 2.1. *Let $\mathbf{p}(t) = (x_1(t), x_2(t), v_1(t), v_2(t))$ be a solution of equation (3) in $\mathcal{U}_\rho$. Then*

(7)

$$t = \pm(F(\alpha_1, x_1(t)) - F(\alpha_1, x_1(0))) = \pm(F(\alpha_2, x_2(t)) - F(\alpha_2, x_2(0)))$$

where $\pm$ indicates that there is a sign that could go either way.

Proof. We have from (4) that

$$\int_0^t dt = \pm \int_{x_1(0)}^{x_1(t)} \frac{dx_1}{\sqrt{2(\alpha_1 + (1/x_1))}} = \pm \int_{x_2(0)}^{x_2(t)} \frac{dx_2}{\sqrt{2(\alpha_2 + (1/x_2))}},$$

from which (7) follows. □

For $t_1, t_2 > 0$, let

$$W_1(t_1) = \{\mathbf{p} = (x_1, x_2, v_1, v_2) \in \mathcal{U}_\rho : \ F(\alpha_1, \ x_1) = t_1\},$$
$$W_2(t_2) = \{\mathbf{p} = (x_1, x_2, v_1, v_2) \in \mathcal{U}_\rho : \ F(\alpha_2, \ x_2) = t_2\}.$$

Corollary 2.1. *Let* $\mathbf{p}(t) = (x_1(t), x_2(t), v_1(t), v_2(t))$ *be a solution of equation* (3) *in* $\mathcal{U}_\rho$. *Then* $x_1(t) \to 0$ *as* $t \to t_1^-$ *if and only if* $\mathbf{p}(0) \in W_1(t_1)$. *Similarly,* $x_2(t) \to 0$ *as* $t \to t_2^-$ *if and only if* $\mathbf{p}(0) \in W_2(t_2)$.

Proof. Observe that $W_1(t_1)$ is defined by $F(\alpha_1, x_1) = t_1$, an equation obtained by letting $t = t_1$, $x_1(t) = 0$ in (7). The $\pm$ sign in (7) is forced to be negative since $dx_1/dt < 0$ as $t \to t_1^-$. The situation for x_2 is similar. □

Let

$$
\begin{aligned}
Y &= F(\alpha_2,\ x_2) - F(\alpha_1,\ x_1) \\
&= \frac{\sqrt{2}}{3} x_2^{3/2} \left[1 + X(\alpha_2, x_2)\right] \\
&\quad - \frac{\sqrt{2}}{3} x_1^{3/2} \left[1 + X(\alpha_1, x_1)\right]
\end{aligned}
\tag{8}
$$

where α_1, α_2 are as in (4) and $X(\alpha, x)$ is as in (5). Y is a crucial new variable. Let us now make the following observations.

(a) Let $\mathbf{p}(t)$ be a given solution of equation (3). Then

$$
\frac{dY}{dt}(\mathbf{p}(t)) = 0.
$$

(b) The algebraic variety defined by $Y(\mathbf{p}) = 0$ and its backward images in time form a co-dimensional one immersed sub-manifold in phase space containing all solutions heading toward simultaneous binary collisions.

(c) We can solve for x_1/x_2 and x_2/x_2 from (8) to obtain

$$
\begin{aligned}
\frac{x_1}{x_2} &= \sqrt[2/3]{\frac{1 + X(\alpha_2, x_2) - [(3\sqrt{2}Y)/(2x_2^{3/2})]}{1 + X(\alpha_1, x_1)}}; \\
\frac{x_2}{x_1} &= \sqrt[2/3]{\frac{1 + X(\alpha_1, x_1) + [(3\sqrt{2}Y)/(2x_1^{3/2})]}{1 + X(\alpha_2, x_2)}}
\end{aligned}
\tag{9}
$$

where $X(\alpha, x)$ is as in (5).

B. A change of variables and the regularized equations. We are now ready to introduce regularization variables. Let us denote the new phase variables as $\mathbf{q} = (\xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, Y)$, and the new time as τ.

First, (ξ_1, η_1) and (ξ_2, η_2) are determined by (x_1, v_1) and (x_2, v_2) through

$$x_1 = \frac{\xi_1^2}{2}, \quad v_1 = \frac{\eta_1}{\xi_1}; \quad x_2 = \frac{\xi_2^2}{2}, \quad v_2 = \frac{\eta_2}{\xi_2}. \tag{10}$$

These are the well known Levi-Civita changes of coordinates. Second, (α_1, α_2) are defined by using (4) and Y by using (8). Third, τ is defined through

$$d\tau = \frac{1}{2}\left(\frac{1}{x_1} + \frac{1}{x_2}\right) dt, \tag{11}$$

and in reverse we have

$$dt = \left(\frac{1}{\xi_1^2} + \frac{1}{\xi_2^2}\right)^{-1} d\tau. \tag{12}$$

The new equations for $\mathbf{q} = (\xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, Y)$ derived from equation (3) are as follows.

$$\frac{d\xi_1}{d\tau} = \frac{1}{1+f_1}\eta_1, \qquad \frac{d\eta_1}{d\tau} = \frac{2}{1+f_1}\alpha_1\xi_1; \tag{13}$$

$$\frac{d\xi_2}{d\tau} = \frac{1}{1+f_2}\eta_2, \qquad \frac{d\eta_2}{d\tau} = \frac{2}{1+f_2}\alpha_2\xi_2; \tag{14}$$

$$\frac{d\alpha_1}{d\tau} = 0, \qquad \frac{d\alpha_2}{d\tau} = 0; \tag{15}$$

$$\frac{dY}{d\tau} = 0. \tag{16}$$

where

$$\begin{aligned} f_1 &= \sqrt[2/3]{\frac{1 + \sum_{n=1}^{\infty}[(3c_n)/(2^n(2n+3))]{\alpha_2}^n\xi_2^{2n} - (6Y/\xi_2^3)}{1 + \sum_{n=1}^{\infty}[(3c_n)/(2^n(2n+3))]\alpha_1^n\xi_1^{2n}}}, \\ f_2 &= \sqrt[2/3]{\frac{1 + \sum_{n=1}^{\infty}[(3c_n)/(2^n(2n+3))]\alpha_1^n\xi_1^{2n} + (6Y/\xi_1^3)}{1 + \sum_{n=1}^{\infty}[(3c_n)/(2^n(2n+3))]\alpha_2^n\xi_2^{2n}}}. \end{aligned} \tag{17}$$

In order for the solutions of the regularized equations (13)–(16) to represent the solutions of equation (3), we also need to impose constraints

$$2\xi_1^2\alpha_1 = \eta_1^2 - 4, \qquad 2\xi_2^2\alpha_2 = \eta_2^2 - 4 \tag{18}$$

and

$$\begin{aligned} Y = &\left[\frac{1}{6} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{c_n}{2^n(n+(3/2))}\alpha_2^n\xi_2^{2n}\right]\xi_2^3 \\ &- \left[\frac{1}{6} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{c_n}{2^n(n+(3/2))}\alpha_1^n\xi_1^{2n}\right]\xi_1^3. \end{aligned} \tag{19}$$

Equation (18) is derived from (4) and (19) from (8).

Derivations of equations (13)–(16). Equations (15) and (16) follow from the fact that α_1, α_2 and Y are first integrals of equation (3). For the first item of equation (13) we differentiate $\xi_1^2 = 2x_1$ to obtain

$$\frac{d\xi_1}{dt} = \frac{v_1}{\xi_1} = \frac{\eta_1}{\xi_1^2}.$$

We have, by using (12),

$$\frac{d\xi_1}{d\tau} = \frac{\eta_1}{1 + (\xi_1^2/\xi_2^2)}.$$

We then substitute f_1 for ξ_1^2/ξ_2^2 using (9).

For the second item of equation (13) we differentiate $\eta_1 = v_1\xi_1$ to obtain

$$\frac{d\eta_1}{dt} = v_1\frac{d\xi_1}{dt} + \xi_1\frac{dv_1}{dt} = \frac{v_1^2}{\xi_1} - \frac{\xi_1}{x_1^2},$$

where equation (3) is used in obtaining the second equality. We then use (4) and (12) to conclude

$$\frac{d\eta}{d\tau} = \frac{2\alpha_1\xi_1}{1 + (\xi_1^2/\xi_2^2)}.$$

We again substitute f_1 for ξ_1^2/ξ_2^2 by using (9). The derivations for equations in (14) are similar. □

Let

$$\mathcal{V}_\rho = \{\mathbf{q} = (\xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, Y) : \ |\alpha_1 \xi_1^2|, |\alpha_2 \xi_2^2| < 2\rho\}$$

be the correspondence of $\mathcal{U}_\rho$ in phase space $\mathbf{q}$, and let $\mathcal{M}_\rho$ be the algebraic variety defined by (18) and (19) in $\mathcal{V}_\rho$. Our next lemma assures that (18) and (19) are nature constraints for equations (13)–(16).

Lemma 2.2. *Let $\mathbf{q}(\tau), \tau \in (\tau_1, \tau_2)$ be a solution of equations (13)–(16) in $\mathcal{V}_\rho$. If $\{\mathbf{q}(\tau), \tau \in (\tau_1, \tau_2)\} \cap \mathcal{M}_\rho \neq \varnothing$, then it is included in $\mathcal{M}_\rho$.*

Proof. Recall that $\mathbf{q} = (\xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, Y)$ are used to denote the new phase variables and $\mathbf{p} = (x_1, x_2, v_1, v_2)$ the old phase variables. Let $\mathbf{q}(\tau)$ be a solution of equations (13)–(16) and assume that $\mathbf{q}(\tau_0)$ satisfies (18) and (19). By using (10) we obtain a corresponding value $\mathbf{p}_0$. Let $\mathbf{p}(t)$ be the solution of equation (3) satisfying $\mathbf{p}(0) = \mathbf{p}_0$. We then use (4), (8), (10) and

$$\tau = \tau_0 + \frac{1}{2} \int_0^t \left(\frac{1}{x_1(t)} + \frac{1}{x_2(t)} \right) dt \tag{20}$$

to convert $\mathbf{p}(t)$ to a function of $\mathbf{q}$ in τ, which we denote as $\widehat{\mathbf{q}}(\tau)$. We caution that there is more than one way to make the last conversion but we can always choose to make $\widehat{\mathbf{q}}(\tau_0) = \mathbf{q}(\tau_0)$. We then observe that

(i) $\widehat{\mathbf{q}}(\tau)$ satisfies (13)–(16) by the derivation in the above, and by uniqueness $\widehat{\mathbf{q}}(\tau) = \mathbf{q}(\tau)$ for all τ;

(ii) on the other hand, $\widehat{\mathbf{q}}(\tau)$ satisfies (18) and (19) by default. □

Equations (13)–(16) *confined on $\mathcal{M}_\rho$ are the regularized equations we seek for* (3). Other solutions of equations (13)–(16) are not relevant.

Remark. It is sometimes helpful to use new equations obtained from (13)–(16) by substitutions derived from constraints (18) and (19). These new equations might look different but, confined on $\mathcal{M}_\rho$, the vector field they define is the same as the one defined by the old equations. For instance, replacing f_1 in equation (13) by ξ_1^2/ξ_2^2 while keeping all the other equations the same would give us a set of equations that looks new but on $\mathcal{M}_\rho$ it is the same as (13)–(16).

C. Regularization result. We are now ready to prove

Theorem 1. *Let* $\mathbf{p}(t)$, $t \in (t_1, t_2)$ *be a solution of equation* (3) *in* $\mathcal{U}_\rho$. *Assume that* $\mathbf{p}(t) \to \Delta$ *as* $t \to t_2^-$. *Let* $\tau(t)$ *be defined by* (20), *and let* $\mathbf{q}(\tau), \tau \in (\tau_1, \tau_2)$ *be the functions obtained from* $\mathbf{p}(t)$ *through* (4), (8) *and* (10). *Then*

(a) $\mathbf{q}(\tau)$ *is a solution of equations* (13)–(16) *on* $\mathcal{M}_\rho$;

(b) $\tau_2 := \tau(t_2) < \infty$, *and* $\mathbf{q}(\tau_2) := \lim_{\tau \to \tau_2^-} \mathbf{q}(\tau)$ *is well-defined*; *and*

(c) *equations* (13)–(16) *defined on* $\mathcal{M}_\rho$ *are real analytic at* $\mathbf{q}(\tau_2)$.

Proof. (a) This follows from the derivations of equations (13)–(16) in subsection 2B. We caution that (10) allows different ways to convert $\mathbf{p}(t)$ to $\mathbf{q}(\tau)$ because ξ_i ($= \pm\sqrt{2x_i}$ by (10)) can assume different signs. This is a well-known characteristic of Levi-Civita variables. For definiteness, let us choose the positive sign so that $\xi_i = \sqrt{2x_i}$. We also note that τ_0 in (20) is arbitrary.

(b) It is well known that, when a collision singularity occurs at t_2,

$$U(t) := \frac{1}{x_1(t)} + \frac{1}{x_2(t)} \sim (t - t_2)^{-2/3}.$$

From this it follows that

$$\tau_2 = \tau_0 + \frac{1}{2} \int_0^{t_2} U(t)\, dt < \infty.$$

Now, for $\mathbf{q}(\tau_2)$: $\alpha_1(\tau_2), \alpha_2(\tau_2)$ and $Y(\tau_2)$ are integral constants determined by initial conditions. Observe that $x_i(t) \to$ a definite limit as $t \to t_2^-$, which we denote as $x_i(t_2)$. We let $\xi_i(\tau_2) = \sqrt{2x_i(t_2)}$. Finally for $\eta_i(\tau_2)$ we use $\eta_i(\tau_2) = v_i(t_2)/\xi_i(\tau_2)$ if $\xi_i(\tau_2) \neq 0$ ($v_i(t) \to v_i(t_2)$ is a definite limit in this case). If $\xi_i(\tau_2) = 0$, then $\eta_i^2(\tau_2) = 4$ according to (18), from which it follows that $\eta_i(\tau_2) = -2$. $\eta_i(\tau_2)$ is negative because we have used a positive sign for $\xi_i(\tau_2)$.

(c) We have three cases to consider depending on what happens at t_2: (1) $x_1(t_2) = 0$ and $x_2(t_2) = 0$; (2) $x_1(t_2) = 0$ but $x_2(t_2) \neq 0$; and (3) $x_2(t_2) = 0$ but $x_1(t_2) \neq 0$. They correspond to the cases of $Y = 0$, $Y > 0$ and $Y < 0$ respectively.

Case $Y = 0$**.** This is the case of simultaneous binary collisions. Set $Y = 0$ in equations (13) and (14). It is clear that the functions on the right hand side are all analytic at the values of $\mathbf{q}(\tau_2)$ given in the above. We conclude that the singularity of simultaneous binary collisions is regularized.

Case $Y < 0$**.** This is a case of binary collision at which $x_2(t_2) = 0$. In this case $\xi_1(\tau_2) \neq 0$, $\xi_2(\tau_2) = 0$. To see that this singularity is removed in the first item of equation (13), we rewrite it as

$$\frac{d\xi_1}{d\tau} = \frac{\eta_1 \xi_2^2}{\xi_2^2 + \sqrt[2/3]{\frac{(2/3)\xi_2^3 + \sum_{n=1}^{\infty}[c_n/(2^n(n+(3/2)))]\alpha_2{}^n \xi_2^{2n+3} - 4Y}{(2/3) + \sum_{n=1}^{\infty}[c_n/(2^n(n+(3/2)))]\alpha_1^n \xi_1^{2n}}}}. \tag{21}$$

It is clear that $\xi_2 = 0$ is not a singularity of the function on the right hand side because $-4Y > 0$ by assumption. The second item is handled similarly.

For the first item of equation (14) we replace f_2 by ξ_2^2/ξ_1^2 to rewrite this equation as

$$\frac{d\xi_2}{d\tau} = \frac{\xi_1^2 \eta_2}{\xi_1^2 + \xi_2^2}.$$

(See the remark we made at the end of subsection 2B.) The function on the right hand side is obviously real analytic at $\mathbf{q}(\tau_2)$ since $\xi_1(\tau_2) \neq 0$. The argument for the second item follows the same line of reasoning.

Case $Y > 0$**.** Similar to the case $Y < 0$. □

Theorem 1 is a precise way to state that all singularities of collision in equation (3) are removed by transferring to equations (13)–(16) on $\mathcal{M}_\rho$.

3. On a restricted four-body problem. In this section we introduce regularization variables for the singularity of simultaneous binary collisions in a restricted four-body problem. New issues arise as we move from the decoupled Kepler problems studied in Section 2 to this restricted gravitational system that is not integrable.

3.1. Equations of motion. We consider gravitational particles m_1, m_2, m_3 and m_4 positioned at $x_1 < x_2 < x_3 < x_4$ respectively in $\mathbf{R}$. In this section we assume $m_1 = m_4 = 0$. To simplify the writing we also assume $m_2 = m_3 = 1$. Our assumptions on m_2 and m_3 are not necessary, and the construction presented in this section applies in principle to arbitrary combinations of positive m_2 and m_3.

Let

$$u_1 = x_2 - x_1, \qquad u_2 = x_4 - x_3, \qquad \widehat{u} = x_3 - x_2$$

and $v_i = du_i/dt$ for $i = 1, 2$, $\widehat{v} = d\widehat{u}/dt$. $(u_1, u_2, \widehat{u}, v_1, v_2, \widehat{v})$ are the phase variables. Let

$$\mathcal{K}(u, \widehat{u}) = \frac{1}{u + \widehat{u}}.$$

We denote

$$\mathcal{K}_1 = \mathcal{K}(u_1, \widehat{u}), \qquad \mathcal{K}_2 = \mathcal{K}(u_2, \widehat{u}), \qquad \widehat{\mathcal{K}} = \mathcal{K}(0, \widehat{u}),$$

and write the equations of motion as

$$\begin{aligned} \frac{du_1}{dt} &= v_1, & \frac{dv_1}{dt} &= -\frac{1}{u_1^2} + \frac{\partial \mathcal{K}_1}{\partial u_1} - \frac{\partial \widehat{\mathcal{K}}}{\partial \widehat{u}}; \\ \frac{du_2}{dt} &= v_2, & \frac{dv_2}{dt} &= -\frac{1}{u_2^2} + \frac{\partial \mathcal{K}_2}{\partial u_2} - \frac{\partial \widehat{\mathcal{K}}}{\partial \widehat{u}}; \\ \frac{d\widehat{u}}{dt} &= \widehat{v}, & \frac{d\widehat{v}}{dt} &= 2\frac{\partial \widehat{\mathcal{K}}}{\partial \hat{u}}. \end{aligned} \tag{22}$$

Let

$$\alpha_1 = \frac{1}{2}v_1^2 - \frac{1}{u_1}, \qquad \alpha_2 = \frac{1}{2}v_2^2 - \frac{1}{u_2}. \tag{23}$$

It follows from (22) that

$$\frac{d\alpha_1}{dt} = v_1\left(\frac{\partial \mathcal{K}_1}{\partial u_1} - \frac{\partial \widehat{\mathcal{K}}}{\partial \widehat{u}}\right), \qquad \frac{d\alpha_2}{dt} = v_2\left(\frac{\partial \mathcal{K}_2}{\partial u_2} - \frac{\partial \widehat{\mathcal{K}}}{\partial \widehat{u}}\right). \tag{24}$$

Remarks. (1) $(u_1, u_2, \hat{u}) \in (\mathbf{R}^2)^+ \times \mathbf{R}^+$ is now the space of positions and $\Delta_{1,2} = \{u_1 = u_2 = 0,\ \widehat{u} \in \mathbf{R}^+\}$ is the singular set for simultaneous binary collisions.

(2) Observe that we would get back to the decoupled Kepler problems in Section 2 by letting $\mathcal{K}(u, \widehat{u}) =$ constant in equation (22).

(3) We intend to follow the ideas developed in Section 2. However, because $\mathcal{K}_i$, $\widehat{\mathcal{K}}$ are non-trivial, α_i are no longer first integrals. Consequently, the correspondence of the new variable Y is much less straight forward to define.

(4) Let us also note that, for the restricted four-body problem introduced above, $(\partial^2 \mathcal{K}_i)/(\partial u_1 \partial u_2) = 0$ by design. The fact that the correspondences of these mixed derivatives are not zero in the full collinear four-body problem will post a major hurdle in similar constructions of regularization variables, as we will see in subsection 3.4.

3.2. Variable Y: formal definition and convergence.

A. Outline of strategy. Let $K > 1$ be fixed and $\rho < (100K^8)^{-2}$ positive. In this section $\mathbf{p} = (u_1, u_2, \widehat{u}, v_1, v_2, \widehat{v})$ are the phase variables and

$$\mathcal{U}_{K,\rho} = \{\mathbf{p} \in (\mathbf{R}^3)^+ \times \mathbf{R}^3 :\ u_1, u_2 < \rho;\ K^{-1} < \widehat{u} < K;\ |\alpha_1|, |\alpha_2| < K\}.$$

We only consider solutions of equation (22) in $\mathcal{U}_{K,\rho}$.

Lemma 3.1. *Let $\mathbf{p}(t)$, $t \in (t_1, t_2)$, be a solution of equation (22) in $\mathcal{U}_{K,\rho}$. Then the limits of $u_i, \widehat{u}, \widehat{v}$ are well defined as $t \to t_2^-$. Furthermore, if $u_i(t) \to u_i(t_2) \neq 0$, then $v_i(t)$ has a well defined limit as $t \to t_2^-$.*

The proof of Lemma 3.1 is well documented. See, for instance, [**7**].

We have from (23) that

$$dt = \pm \frac{du_1}{\sqrt{2\,(\alpha_1 + (1/u_1))}}.$$

Let $\mathbf{p}(t) \subset \mathcal{U}_{K,\rho}$ be a solution of equation (22). Integrating on both sides we obtain

$$t - t_1 = \pm \int_0^{u_1(t)} \frac{du_1}{\sqrt{2\left(\alpha_1 + (1/u_1)\right)}} \tag{25}$$

where t_1 is such that $u_1(t_1) = 0$. Let us denote

$$F := \int_0^{u_1} \frac{du_1}{\sqrt{2\left(\alpha_1 + (1/u_1)\right)}}.$$

Since α_1 is no longer a first integral of equation (22), F as written above is not precisely a well-defined definite integral. Let us, however, put this subtlety aside for now and treat F formally as if it is well defined. We then expand the integrand to obtain

$$F = \frac{1}{\sqrt{2}}\left[\frac{2}{3}u_1^{3/2} + \sum_{n=1}^{\infty} c_n \int_0^{u_1} \alpha_1^n u_1^{n+(1/2)} du_1\right] \tag{26}$$

where $c_n, n > 0$ are as in (6) in Section 2.

To each of the integrals in (26) (as well as the new ones we will soon encounter), a *degree* is assigned according to the power of u_1 in the integrand. For instance, the integral

$$I_n = \int_0^{u_1} \alpha_1^n u_1^{n+(1/2)} du_1$$

is an integral of degree $n + (1/2)$. Our strategy is to use integration by parts together with equation (22) to replace all integrals in (26) with integrals of degrees higher and higher to eventually write F explicitly in phase variables. Let us take I_n as an example. We have

$$\begin{aligned}
I_n &= \frac{1}{n+(3/2)} \int_0^{u_1} \alpha_1^n du_1^{n+(3/2)} \\
&= \frac{1}{n+(3/2)} \alpha_1^n u_1^{n+(3/2)} \\
&\quad - \frac{n}{n+(3/2)} \int_0^{u_1} \alpha_1^{n-1} \frac{d\alpha_1}{du_1} u_1^{n+(3/2)} du_1 \\
&= \frac{1}{n+(3/2)} \alpha_1^n u_1^{n+(3/2)} - \frac{n}{n+(3/2)} \int_0^{u_1} \alpha_1^{n-1} \frac{\partial \mathcal{K}_1}{\partial u_1} u_1^{n+(3/2)} du_1 \\
&\quad + \frac{n}{n+(3/2)} \int_0^{u_1} \alpha_1^{n-1} \frac{\partial \widehat{\mathcal{K}}}{\partial \widehat{u}} u_1^{n+(3/2)} du_1
\end{aligned}$$

where for the last equality we replaced $d\alpha_1/du_1$ by using (24). I_n is then the summation of a term that is explicit in u_1 and α_1 and two integrals of one degree higher.

We now go one step further to transfer the new integrals obtained in the above to integrals of degree even higher. We have for instance

$$\begin{aligned} I &:= \int_0^{u_1} \alpha_1^{n-1} \frac{\partial \mathcal{K}_1}{\partial u_1} u_1^{n+(3/2)} du_1 \\ &= \frac{1}{n+(5/2)} \int_0^{u_1} \alpha_1^{n-1} \frac{\partial \mathcal{K}_1}{\partial u_1} du_1^{n+(5/2)} \\ &= \frac{1}{n+(5/2)} \alpha_1^{n-1} \frac{\partial \mathcal{K}_1}{\partial u_1} u_1^{n+(5/2)} \\ &\quad - \frac{n-1}{n+(5/2)} \int_0^{u_1} \alpha_1^{n-2} \frac{\partial \mathcal{K}_1}{\partial u_1} u_1^{n+(5/2)} \frac{d\alpha_1}{du_1}\, du_1 \\ &\quad - \frac{1}{n+(5/2)} \int_0^{u_1} \alpha_1^{n-1} \frac{\partial^2 \mathcal{K}_1}{\partial u_1^2} u_1^{n+(5/2)}\, du_1 \\ &\quad - \frac{1}{n+(5/2)} \int_0^{u_1} \alpha_1^{n-1} \frac{\partial^2 \mathcal{K}_1}{\partial u_1 \partial \widehat{u}} u_1^{n+(5/2)} \frac{\widehat{v}}{v_1}\, du_1. \end{aligned}$$

The first two integrals can be further converted to integrals of one degree higher the same way. The last one, however, is with a new factor $\widehat{v}v_1^{-1}$. We will keep $\widehat{v}$, which is bounded, therefore harmless, but rewrite v_1^{-1} through (23) as

$$v_1^{-1} = \frac{1}{\sqrt{2}}(u_1^{1/2} + \sum_{n=1}^{\infty} c_n \alpha_1^n u_1^{n+(1/2)}) \tag{27}$$

where c_n is as in (6). The third integral is then replaced by a sequence of integrals of ascending degrees through (27).

Based on computations of a similar nature, we now proceed as follows. Let us start with (26). First we replace the integral of degree 3/2 in (26) by a function written explicitly in phase variables and a number of integrals of higher degree. We then move up to replace all integrals of degree 5/2 the same way and so on.[2] This process goes forever and, at the end, we will be able to write F explicitly as a function of u_1, α_1, $\widehat{u}$ and $\widehat{v}$. Let us also remember that, for the replacement process described above to be meaningful, the infinite series we obtain at the end must converge.

B. A formal inductive process. We now formally introduce a replacement process that is convergent following the strategy outlined in subsection 3.2A for

$$F = \frac{1}{\sqrt{2}} \int_0^{u_1} \frac{du_1}{\sqrt{\alpha_1 + (1/u_1)}}.$$

Initially, we let

$$F = \mathcal{F}^{(3)} := \frac{1}{\sqrt{2}} \left[\frac{2}{3} u_1^{3/2} + \sum_{n=1}^{\infty} \int_0^{u_1} c_n \alpha_1^n u_1^{n+(1/2)} du_1 \right]. \tag{28}$$

Proposition 3.1. *Let $m \geq 3$. We have*

$$\begin{aligned} F = \mathcal{F}^{(m)} := & \sum_{n=3}^{m+1} \Bigg(\sum_{j \leq \widehat{S}(m,n)} \widehat{f}_j^{(n)}(u_1, \alpha_1, \widehat{u}, \widehat{v}) \Bigg) u_1^{n/2} \\ & + \sum_{n=m}^{\infty} \Bigg(\sum_{j \leq S(m,n)} \int_0^{u_1} f_j^{(n)}(u_1, \alpha_1, \widehat{u}, \widehat{v}) u_1^{n/2} du_1 \Bigg) \end{aligned} \tag{29}$$

where

(a) *on non-integral terms*:

(i) for every $j \leq \widehat{S}(m,n)$, there exist coefficients $\widehat{C}_{n,j}$ satisfying $|\widehat{C}_{n,j}| < 10^n$ and non-negative integers i_k for $k = 1$ to 4 satisfying $i_k \leq 2n$ such that

$$\widehat{f}_j^{(n)}(u_1, \alpha_1, \widehat{u}, \widehat{v}) = \widehat{C}_{n,j} \alpha_1^{i_1} \widehat{v}^{i_2} (u_1 + \widehat{u})^{-i_3} \widehat{u}^{-i_4};$$

(ii) $\widehat{S}(m, m+1) < 5^{m-1}$;

(b) *on integrals*:

(i) for every $j \leq S(m,n)$, there exist coefficients $C_{n,j}$ satisfying $|C_{n,j}| < 10^n$ and non-negative integers i_k for $k = 1$ to 4 satisfying $i_k \leq 2n$, such that

$$f_j^{(n)}(u_1, \alpha_1, \widehat{u}, \widehat{v}) = C_{n,j} \alpha_1^{i_1} \widehat{v}^{i_2} (u_1 + \widehat{u})^{-i_3} \widehat{u}^{-i_4};$$

(ii) $S(m,m) < 5^m$.

Remark. $\mathcal{F}^{(m)}$ in (29) represents the integral F obtained at the end of stage m of a replacement process we will introduce momentarily in the proof of Proposition 3.1. According to (29) and (a) (i), the non-integral part is a finite sum of terms of ascending degrees in u_1, the highest of which is $(m+1)/2$. The term of degree $n/2$, $n \leq m+1$, in this summation is in turn a summation of $\widehat{S}(m,n)$ terms, each of which is in the form assumed in Proposition (a) (i). Similarly, according to (29) and (b) (i), the integral part is a series of integrals of ascending degrees, the lowest of which is $m/2$. We have in total $S(m,n)$ integrals of degree $n/2$ for $n \geq m$, each of which is in the form assumed in b (i). (a) (ii) and b (ii) claim that the growth of the number of terms created by replacement is slower than exponential, a crucial fact for convergence. Let us also note that the increment of power in u_1 is half instead of one in $\mathcal{F}^{(m)}$ because of the use of (27). However, through integration by parts, the non-integral terms obtained from an integral of degree $m/2$ is of degree $(m/2)+1$. This is why the non-integral part is a summation up to $n = m+1$ instead of $n = m$.

Proof of Proposition 3.1. First we prove (a) (i), (b) (i) and (29) inductively. For $m = 3$, $\mathcal{F}^{(3)}$ is as in (28). It is obviously in the form assumed by (29) satisfying Proposition 3.1 a (i) and b (i).

Let us now inductively assume that $\mathcal{F}^{(m)}$, $m = 3, \ldots, M$ are well defined in the form assumed by (29) and Proposition 3.1 a (i) and b (i) hold up to $m = M$. $\mathcal{F}^{(M+1)}$ is derived by replacing all integrals of degree $M/2$ in $\mathcal{F}^{(M)}$ as follows. Let

$$I = \int_0^{u_1} f_j^{(M)}(u_1, \alpha_1, \widehat{u}, \widehat{v}) u_1^{M/2} du_1$$

be an integral of degree $M/2$ in $\mathcal{F}^{(M)}$,

$$\begin{aligned} I &= C_{M,j} \int_0^{u_1} \alpha_1^{i_1} \widehat{v}^{i_2} (u_1 + \widehat{u})^{-i_3} \widehat{u}^{-i_4} u_1^{M/2} du_1 \\ &= \frac{C_{M,j}}{(M/2)+1} \int_0^{u_1} \alpha_1^{i_1} \widehat{v}^{i_2} (u_1 + \widehat{u})^{-i_3} \widehat{u}^{-i_4} du_1^{(M/2)+1} \\ &= \frac{C_{M,j}}{(M/2)+1} \Big(\alpha_1^{i_1} \widehat{v}^{i_2} (u_1 + \widehat{u})^{-i_3} \widehat{u}^{-i_4} u_1^{(M/2)+1} \\ &\qquad\qquad - \int_0^{u_1} u_1^{(M/2)+1} d\left(\alpha_1^{i_1} \widehat{v}^{i_2} (u_1 + \widehat{u})^{-i_3} \widehat{u}^{-i_4}\right) \Big). \end{aligned}$$

Hence

$$I = \frac{C_{M,j}}{(M/2)+1}\alpha_1^{i_1}\widehat{v}^{i_2}(u_1+\widehat{u})^{-i_3}\widehat{u}^{-i_4}u_1^{(M/2)+1} - \sum_{i=1}^{4}\mathcal{I}_i. \tag{30}$$

The first term on the right hand side is the contribution of I to the non-integral part of $\mathcal{F}^{(M+1)}$. To be more precise we get one $\widehat{f}_j^{(M+1)}$ from I such that

$$\widehat{f}_j^{(M+1)} = \frac{C_{M,j}}{(M/2)+1}\alpha_1^{i_1}\widehat{v}^{i_2}(u_1+\widehat{u})^{-i_3}\widehat{u}^{-i_4}.$$

For this $\widehat{f}_j^{(M+1)}$ term Proposition 3.1 a (i) is satisfied with

$$\widehat{C}_{M+1,j} = \frac{C_{M,j}}{(M/2)+1}.$$

We also have in (30)

$$\mathcal{I}_1 = \frac{i_1 C_{M,j}}{(M/2)+1}\int_0^{u_1}\alpha_1^{i_1-1}\widehat{v}^{i_2}(u_1+\widehat{u})^{-i_3}\widehat{u}^{-i_4}\frac{d\alpha_1}{du_1}u_1^{(M/2)+1}du_1,$$
$$\mathcal{I}_2 = \frac{i_2 C_{M,j}}{(M/2)+1}\int_0^{u_1}\alpha_1^{i_1}\widehat{v}^{i_2-1}(u_1+\widehat{u})^{-i_3}\widehat{u}^{-i_4}\frac{d\widehat{v}}{dt}\frac{1}{v_1}u_1^{(M/2)+1}du_1,$$
$$\mathcal{I}_3 = -\frac{i_3 C_{M,j}}{(M/2)+1}\int_0^{u_1}\alpha_1^{i_1}\widehat{v}^{i_2}(u_1+\widehat{u})^{-i_3-1}\widehat{u}^{-i_4}\left(1+\frac{\widehat{v}}{v_1}\right)u_1^{(M/2)+1}du_1,$$
$$\mathcal{I}_4 = -\frac{i_4 C_{M,j}}{(M/2)+1}\int_0^{u_1}\alpha_1^{i_1}\widehat{v}^{i_2}(u_1+\widehat{u})^{-i_3}\widehat{u}^{-i_4-1}\frac{\widehat{v}}{v_1}u_1^{(M/2)+1}du_1.$$

All these integrals are to be further transformed as follows.

(a) *On* $\mathcal{I}_1$: From (24) we have

(31)
$$\begin{aligned}\mathcal{I}_1 &= \frac{i_1 C_{M,j}}{(M/2)+1}\int_0^{u_1}\alpha_1^{i_1-1}\widehat{v}^{i_2}(u_1+\widehat{u})^{-i_3}\\ &\qquad\times\widehat{u}^{-i_4}(-(u_1+\widehat{u})^{-2}+(\widehat{u})^{-2})u_1^{(M/2)+1}du_1\\ &= -\frac{i_1 C_{M,j}}{(M/2)+1}\int_0^{u_1}\alpha_1^{i_1-1}\widehat{v}^{i_2}(u_1+\widehat{u})^{-(i_3+2)}\widehat{u}^{-i_4}u_1^{(M/2)+1}du_1\\ &\quad+\frac{i_1 C_{M,j}}{(M/2)+1}\int_0^{u_1}\alpha_1^{i_1-1}\widehat{v}^{i_2}(u_1+\widehat{u})^{-i_3}\widehat{u}^{-(i_4+2)}u_1^{(M/2)+1}du_1.\end{aligned}$$

Hence $\mathcal{I}_1$ is the sum of two integrals of degree $(M/2)+1$, each of which satisfies Proposition 3.1 b (i): note that the degree of these new integrals is $(M/2)+1$ and recall that we have assumed inductively that

$$C_{M,j} \leq 10^M, \qquad i_1,\ i_2,\ i_3,\ i_4 \leq 2M,$$

from which it follows that

$$\frac{i_1 C_{M,j}}{(M/2)+1} \leq 10^{M+2}, \qquad i_1 - 1,\ i_2,\ i_3 + 2,\ i_4 + 2 \leq 2(M+2).$$

(b) *On* $\mathcal{I}_2$: Note that $d\widehat{v}/dt = -2\widehat{u}^{-2}$ and, for v_1^{-1}, we use (27). We have

$$\begin{aligned} \mathcal{I}_2 &= \frac{2i_2 C_{M,j}}{(M/2)+1} \int_0^{u_1} \alpha_1^{i_1} \widehat{v}^{i_2-1} (u_1 + \widehat{u})^{-i_3} \widehat{u}^{-(i_4+2)} v_1^{-1} u_1^{(M/2)+1} du_1 \\ &= \frac{\sqrt{2} i_2 C_{M,j}}{(M/2)+1} \sum_{k=0}^{\infty} \int_0^{u_1} c_k \alpha_1^{i_1+k} \widehat{v}^{i_2-1} (u_1+\widehat{u})^{-i_3} \widehat{u}^{-(i_4+2)} u_1^{(M/2)+k+(3/2)} du_1. \end{aligned} \tag{32}$$

Hence $\mathcal{I}_2$ is a summation of infinitely many integrals of ascending degrees, each of which again satisfies Proposition 3.1 b (i). Observe that, to any given $n > M$, $\mathcal{I}_2$ contains only one integral of degree $n/2$.

(c) *On* $\mathcal{I}_3$: Similarly we have

$$\begin{aligned} \mathcal{I}_3 = &- \frac{i_3 C_{M,j}}{(M/2)+1} \int_0^{u_1} \alpha_1^{i_1} \widehat{v}^{i_2} (u_1 + \widehat{u})^{-i_3-1} \\ &\qquad\qquad \times \widehat{u}^{-i_4} u_1^{(M/2)+1} du_1 \\ &- \frac{\sqrt{2} i_3 C_{M,j}}{M+2} \sum_{k=0}^{\infty} \int_0^{u_1} c_k \alpha_1^{i_1+k} \widehat{v}^{i_2+1} (u_1 + \widehat{u})^{-i_3-1} \\ &\qquad\qquad \times \widehat{u}^{-i_4} u_1^{(M/2)+k+(3/2)} du_1. \end{aligned} \tag{33}$$

$\mathcal{I}_3$ is again an infinite summation of integrals of ascending degrees, each of which satisfies Proposition 3.1 b (i). Again, for any given $n > M$, $\mathcal{I}_3$ contains at most one integral of degree $n/2$.

(d) *On* $\mathcal{I}_4$: Similarly we have

$$\mathcal{I}_4 = -\frac{\sqrt{2} i_4 C_{M,j}}{M+2} \sum_{k=0}^{\infty} \int_0^{u_1} c_k \alpha_1^{i_1+k} \widehat{v}^{i_2+1} (u_1 + \widehat{u})^{-i_3} \widehat{u}^{-i_4-1} u_1^{(M/2)+k+(3/2)} du_1. \tag{34}$$

This is similar to $\mathcal{I}_2$ and $\mathcal{I}_3$.

We are now ready to define $\mathcal{F}^{(M+1)}$. For every integral I of degree $M/2$ in $\mathcal{F}^{(M)}$, we replace I by using (30)–(34). This proves Proposition 3.1 a (i), b (i) and (29).

For Proposition 3.1 a (ii) and b (ii) we observe that from (a)–(d) above

Lemma 3.2. (1) $\widehat{S}(M, n) = \widehat{S}(M+1, n)$ *for* $n \leq M+1$,
(2) $\widehat{S}(M+1, M+2) = S(M, M)$,
(3) $S(M+1, n) \leq S(M, n) + 4S(M, M)$ *for* $n > M$.

Proof. From $\mathcal{F}^{(M)}$ to $\mathcal{F}^{(M+1)}$, non-integral terms of degree $< (M+2)/2$ in u_1 are not affected so (1) holds. (2) follows from the observation that every integral I of degree $M/2$ in $\mathcal{F}^{(M)}$ contributes exactly one non-integral term (see (30)) of power $(M/2)+1$ in u_1 to $\mathcal{F}^{(M+1)}$. (3) follows from the fact that, for any $n > M$ fixed, replacing an integral I of degree $M/2$ in $\mathcal{F}^{(M)}$ by using (30) and (a)–(d) above adds at most four more integrals of degree $n/2$ to the previous collection of integrals of the same degree in $\mathcal{F}^{(M)}$. □

We now use Lemma 3.2 (3) inductively to prove Proposition 3.1 (ii). Note that Lemma 3.2 (3) holds for all $M \geq 3$. We have

$$\begin{aligned} S(M+1, M+1) &\leq S(M, M+1) + 4S(M, M) \\ &\leq S(M-1, M+1) + 4S(M-1, M-1) \\ &\quad + 4S(M, M) \\ &\leq S(3, M+1) + 4 \sum_{n=3}^{M} S(n, n). \end{aligned}$$

Note that $S(3, M+1) = 1$. We have

$$S(M+1, M+1) \leq 4S(M, M) + 4S(M-1, M-1) + \cdots + 4S(3,3) + 1 \tag{35}$$

for all $M \geq 3$. Using (35) inductively we obtain

$$S(M, M) \leq 5^M,$$

from which it also follows that

$$\widehat{S}(M+1, M+2) \leq S(M, M) < 5^M.$$

Here Lemma 3.2 (2) is used in obtaining the first inequality. This finishes our proof of Proposition 3.1 (a) (ii) and b (ii). □

Finally we let

$$F(u_1, \alpha_1, \widehat{u}, \widehat{v}) = \frac{\sqrt{2}}{3} u_1^{3/2} + \sum_{n=4}^{\infty} \left(\sum_{\widehat{S}(n,n+1)} \widehat{f}_j^{(n+1)}(u_1, \alpha_1, \widehat{u}, \widehat{v}) \right) u_1^{(n+1)/2}. \tag{36}$$

Proposition 3.2. *Under the assumption that* $\rho < (100K^8)^{-2}$, $F(u_1, \alpha_1, \widehat{u}, \widehat{v})$ *in* (36) *is convergent.*

Proof. From Proposition 3.1 a (i) we have for every j,

$$\widehat{f}_j^{(n)} = \widehat{C}_{n,j} \alpha_1^{i_1} \widehat{v}^{i_2} (u_1 + \widehat{u})^{-i_3} \widehat{u}^{-i_4} u_1^{n/2},$$

from which it follows that

$$|\widehat{f}_j^{(n)}| < 10^n K^{8n} \rho^{n/2} \tag{37}$$

on $\mathcal{U}_{K,\rho}$. Combining Proposition 3.1 a (ii), b (ii) and (37) we have

$$|F(u_1, \alpha_1, \widehat{u}, \widehat{v})| < \sum_{n=3}^{\infty} (100K^8)^n \rho^{n/2}.$$

Hence F converges provided that $\rho < (100K^8)^{-2}$. This proves Proposition 3.2. □

Remark. Let us caution that $F_1 = F(u_1, \alpha_1, \widehat{u}, \widehat{v})$ is not analytic in u_1 at $u_1 = 0$ because the power of u_1 ascends by half instead of one. To get analyticity, we need to replace u_1 by a new variable ξ_1 through $\xi_1^2 = u_1$. F_1 is then analytic in ξ_1 at $\xi_1 = 0$.

C. The new variable Y. Let $F(u_1, \alpha_1, \widehat{u}, \widehat{v})$ be as in (36) and

$$Y := F(u_1, \alpha_1, \widehat{u}, \widehat{v}) - F(u_2, \alpha_2, \widehat{u}, \widehat{v}). \tag{38}$$

We claim that Y is a first integral of equation (22). This claim is proved as follows. For a given solution $\mathbf{p}(t)$ of equation (22) in $\mathcal{U}_{K,\rho}$, let t_1 be such that $u_1(t_1) = 0$ and t_2 such that $u_2(t_2) = 0$. We have from the way $F(u_1, \alpha_1, \widehat{u}, \widehat{v})$ is defined that

$$\begin{aligned} t - t_1 &= F(u_1(t), \alpha_1(t), \widehat{u}(t), \widehat{v}(t)), \\ t - t_2 &= F(u_2(t), \alpha_2(t), \widehat{u}(t), \widehat{v}(t)), \end{aligned} \tag{39}$$

from which we obtain

$$Y(t) = t_2 - t_1.$$

It then follows that

$$\frac{dY}{dt} = 0. \tag{40}$$

Let

$$f(u_1, \alpha_1, \widehat{u}, \widehat{v}) = \frac{3\sqrt{2}}{2} \sum_{n=4}^{\infty} \left(\sum_{\widehat{S}(n,n+1)} \widehat{f}_j^{(n+1)}(u_1, \alpha_1, \widehat{u}, \widehat{v}) \right) u_1^{(n-2)/2}. \tag{41}$$

We have

$$F(u_1, \alpha_1, \widehat{u}, \widehat{v}) = \frac{\sqrt{2}}{3} u_1^{3/2} (1 + f(u_1, \alpha_1, \widehat{u}, \widehat{v})). \tag{42}$$

Note that $f(u_1, \alpha_1, \widehat{u}, \widehat{v})$ is again in a form of power series in $\sqrt{u_1}$ and $f(0, \alpha_1, \widehat{u}, \widehat{v}) = 0$. From (36) and (38) we obtain

$$\frac{u_1}{u_2} = \sqrt[2/3]{\frac{1 + f(u_2, \alpha_2, \widehat{u}, \widehat{v}) + (3\sqrt{2}Y)/(2u_2^{3/2})}{1 + f(u_1, \alpha_1, \widehat{u}, \widehat{v})}} \tag{43}$$

and

$$\frac{u_2}{u_1} = \sqrt[2/3]{\frac{1 + f(u_1, \alpha_1, \widehat{u}, \widehat{v}) - (3\sqrt{2}Y)/(2u_1^{3/2})}{1 + f(u_2, \alpha_2, \widehat{u}, \widehat{v})}}. \tag{44}$$

(43) and (44) are the correspondence of (9) in Section 2.

3.3. Variables of regularization. This subsection is in parallel to subsections 2B and 2C. Let us denote the regularization variables as $\mathbf{q} = (\xi_1, \xi_2, \eta_1, \eta_2, \widehat{u}, \widehat{v}, \alpha_1, \alpha_2, Y)$ and the new time as τ. (ξ_1, η_1) and (ξ_2, η_2) are again determined by (u_1, v_1) and (u_2, v_2) through a Levi-Civita change of coordinates

$$u_1 = \frac{\xi_1^2}{2}, \quad v_1 = \frac{\eta_1}{\xi_1}; \quad u_2 = \frac{\xi_2^2}{2}, \quad v_2 = \frac{\eta_2}{\xi_2}; \tag{45}$$

$\widehat{u}, \widehat{v}$ remain the same, (α_1, α_2) are defined by using (23), and Y is defined by using (38). The new time τ is defined through

$$\tau = \tau_0 + \frac{1}{2}\int_0^t \left(\frac{1}{u_1} + \frac{1}{u_2}\right) dt, \tag{46}$$

and in reverse we have

$$dt = \left(\frac{1}{\xi_1^2} + \frac{1}{\xi_2^2}\right)^{-1} d\tau. \tag{47}$$

The new equations for $\mathbf{q} = (\xi_1, \xi_2, \eta_1, \eta_2, \widehat{u}, \widehat{v}, \alpha_1, \alpha_2, Y)$ derived from equation (22) are as follows.

(48)

$$\frac{d\xi_1}{d\tau} = \frac{1}{1+X_1}\eta_1, \quad \frac{d\eta_1}{d\tau} = \frac{1}{1+X_1}\left(2\alpha_1\xi_1 - \frac{\xi_1^3}{((1/2)\xi_1^2+\widehat{u})^2} + \frac{\xi_1^3}{\widehat{u}^2}\right);$$
$$\frac{d\xi_2}{d\tau} = \frac{1}{1+X_2}\eta_2, \quad \frac{d\eta_2}{d\tau} = \frac{1}{1+X_2}\left(2\alpha_2\xi_2 - \frac{\xi_2^3}{(\frac{1}{2}\xi_2^2+\widehat{u})^2} + \frac{\xi_2^3}{\widehat{u}^2}\right);$$
$$\frac{d\widehat{u}}{d\tau} = \frac{1}{1+X_1}\xi_1^2\widehat{v}, \quad \frac{d\widehat{v}}{d\tau} = \frac{-1}{1+X_1}\frac{\xi_1^2}{\widehat{u}^2};$$
$$\frac{d\alpha_1}{d\tau} = \frac{1}{1+X_1}\xi_1\eta_1\left(-\frac{1}{((1/2)\xi_1^2+\widehat{u})^2} + \frac{1}{\widehat{u}^2}\right),$$
$$\frac{d\alpha_2}{d\tau} = \frac{1}{1+X_2}\xi_2\eta_2\left(-\frac{1}{((1/2)\xi_2^2+\widehat{u})^2} + \frac{1}{\widehat{u}^2}\right);$$
$$\frac{dY}{d\tau} = 0$$

where

$$X_1 = \sqrt[2/3]{\frac{1+f((1/2)\xi_2^2,\alpha_2,\widehat{u},\widehat{v}) + (6Y/\xi_2^3)}{1+f((1/2)\xi_1^2,\alpha_1,\widehat{u},\widehat{v})}},$$
$$X_2 = \sqrt[2/3]{\frac{1+f((1/2)\xi_1^2,\alpha_1,\widehat{u},\widehat{v}) - (6Y/\xi_1^3)}{1+f((1/2)\xi_2^2,\alpha_2,\widehat{u},\widehat{v})}}$$

and f is as in (41). For $i = 1$ and 2, $f((1/2)\xi_i^2, \alpha_i, \widehat{u}, \widehat{v}, Y)$ is analytic in ξ_i at $\xi_i = 0$. We also have $f(0, \alpha_i, \widehat{u}, \widehat{v}) = 0$.

The following constraints are further imposed on equation (48):

(49)
$$2\xi_1^2\alpha_1 = \eta_1^2 - 4, \qquad 2\xi_2^2\alpha_2 = \eta_2^2 - 4;$$

(50)
$$Y = \frac{1}{6}\xi_1^3\left(1 + f\left(\frac{1}{2}\xi_1^2, \alpha_1, \widehat{u}, \widehat{v}\right)\right) - \frac{1}{6}\xi_2^3\left(1 + f\left(\frac{1}{2}\xi_2^2, \alpha_2, \widehat{u}, \widehat{v}\right)\right).$$

The derivations of equation (48) and the constraints (49) and (50) are straightforward, which we leave to the reader.

Let $\mathcal{V}_{K,\rho}$ be the correspondence of $\mathcal{U}_{K,\rho}$ in new phase variables and $\mathcal{M}_{K,\rho}$ the algebraic variety defined by (49) and (50) in $\mathcal{V}_{K,\rho}$. Let $\mathbf{q}(\tau)$, $\tau \in (\tau_1, \tau_2)$, be a solution of equation (48) in $\mathcal{V}_{K,\rho}$. We claim once more that, if $\{\mathbf{q}(\tau), \tau \in (\tau_1, \tau_2)\} \cap \mathcal{M}_{K,\rho} \neq \varnothing$, then it is included in $\mathcal{M}_{K,\rho}$. Our proof for this last claim is identical to that of Lemma 2.2 in Section 2. *Equation* (48) *defined on* $\mathcal{M}_{K,\rho}$ *is our regularized equation for* (22). In parallel to Theorem 1 in Section 2, we have

Theorem 2. *Let* $\mathbf{p}(t)$, $t \in (t_1, t_2)$, *be a solution of equation* (22) *in* $\mathcal{U}_{K,\rho}$. *Assume that* $\mathbf{p}(t) \to \Delta$ *as* $t \to t_2^-$. *Let* $\tau(t)$ *be defined by* (46) *and* $\mathbf{q}(\tau), \tau \in (\tau_1, \tau_2)$ *the functions obtained from* $\mathbf{p}(t)$ *through* (23), (38) *and* (45). *Then*:

(a) $\mathbf{q}(\tau)$ *is a solution of equation* (48) *on* $\mathcal{M}_{K,\rho}$;

(b) $\tau_2 := \tau(t_2) < \infty$, *and* $\mathbf{q}(\tau_2) := \lim_{\tau \to \tau_2^-} \mathbf{q}(\tau)$ *is well-defined*; *and*

(c) *equation* (48) *defined on* $\mathcal{M}_{K,\rho}$ *is real analytic at* $\mathbf{q}(\tau_2)$.

The proof of this theorem is completely parallel to that of Theorem 1 in Section 2. We again leave the details to the reader.

3.4. On the collinear four-body problem. Let us now consider equation (1) in Section 1 for the collinear four-body problem assuming $x_1 < x_2 < x_2 < x_4$. To avoid messy writings we set $m_1 = m_2 = m_3 = m_4 = 1$. We also assume that the center of masses of the four bodies are fixed at $x = 0$, i.e.,

$$\sum_{i=1}^{4} x_i = 0, \qquad \sum_{i=1}^{4} \frac{dx_i}{dt} = 0. \tag{51}$$

This helps in cutting the dimension of the phase space down by two. Let

$$u_1 = x_2 - x_1, \quad u_2 = x_4 - x_3, \quad \widehat{u}_1 = x_1 + x_2, \quad \widehat{u}_2 = x_3 + x_4, \tag{52}$$

and $v_i = (du_i/dt)$, $\widehat{v}_i = (d\widehat{u}_i/dt)$ for $i = 1, 2$. Note that by (51) $\widehat{u}_1 = -\widehat{u}_2$, $\widehat{v}_1 = -\widehat{v}_2$ so there are only six independent variables out of $(u_i, \widehat{u}_i, v_i, \widehat{v}_i)$, $i = 1, 2$. We denote $\widehat{u} = \widehat{u}_1, \widehat{v} = \widehat{v}_1$ and use

$\mathbf{p} = (u_1, u_2, \widehat{u}, v_1, v_2, \widehat{v})$ to rewrite equation (1) as

$$\begin{aligned} \frac{du_1}{dt} &= v_1, & \frac{dv_1}{dt} &= -\frac{2}{u_1^2} + 2\frac{\partial \mathcal{K}}{\partial u_1}; \\ \frac{du_2}{dt} &= v_2, & \frac{dv_2}{dt} &= -\frac{2}{u_2^2} + 2\frac{\partial \mathcal{K}}{\partial u_2}; \\ \frac{d\hat{u}}{dt} &= \widehat{v}, & \frac{d\widehat{v}}{dt} &= 2\frac{\partial \mathcal{K}}{\partial \widehat{u}} \end{aligned} \tag{53}$$

where

$$\begin{aligned} \mathcal{K} = &\frac{2}{-2\widehat{u} + u_2 - u_1} + \frac{2}{-2\widehat{u} + u_2 + u_1} \\ &+ \frac{2}{-2\widehat{u} - u_2 - u_1} + \frac{2}{-2\widehat{u} - u_2 + u_1}. \end{aligned} \tag{54}$$

$(u_1, u_2, \widehat{u}) \in (\mathbf{R}^2)^+ \times \mathbf{R}^-$ is now the space of positions and $\Delta_{12,34} = \{u_1 = u_2 = 0, \widehat{u} \in \mathbf{R}^-\}$ is the singular set for simultaneous binary collisions. The potential function in $(u_1, u_2, \widehat{u}_1)$ is $U = (1/u_1) + (1/u_2) + \mathcal{K}$ and $\mathcal{K}(u_1, u_2, \widehat{u}_1)$ is analytic on $\Delta_{12,34}$. Again, let $\alpha_1 := (1/4)v_1^2 - (1/u_1)$, $\alpha_2 := (1/4)v_2^2 - (1/u_2)$. We have

$$\frac{d\alpha_1}{dt} = v_1 \frac{\partial \mathcal{K}}{\partial u_1}, \qquad \frac{d\alpha_2}{dt} = v_2 \frac{\partial \mathcal{K}}{\partial u_2}. \tag{55}$$

As we apply the replacement strategy of subsection 3.2 to the collinear four-body problem, a technical difficulty caused by interactions between colliding pairs occurs. Let us explain in detail.

We follow the same strategy aiming at writing the integral F explicitly in phase variables through inductive use of integration by part and equations (53) and (55). Let us start again from (26) in subsection 3.2 and repeat the computation appearing in subsection 3.2A for I_n. We replace $d\alpha_1/du_1$ by $\partial\mathcal{K}/\partial u_1$ according to (55). One of the new integrals we obtain is in the form of

$$\int_0^{u_1} \alpha_1^{n-1} \frac{\partial^2 \mathcal{K}}{\partial u_1 \partial u_2} \frac{v_2}{v_1} u_1^{n+(5/2)} du_1$$

due to the fact that $(\partial^2\mathcal{K})/(\partial u_1 \partial u_2) \neq 0$. A simple computation shows that the degree of this integral will never go up (nor it will go down) by a direct combination of integration by part and equation (53).

The true implication of this technical difficulty is not at all clear to us. It is entirely possible that this is merely a resolvable technical issue, although at the moment we do not know how to resolve it. It is also conceivable that this is an intrinsic difficulty that occurred only because what we have aimed to construct is not at all in existence for the collinear four-body problem. In any case, it is evident that, as far as the issue of regularization transforms is concerned, *the singularity of simultaneous binary collisions is in fact very different from that of two independent binary collisions* and the difference is due to the existence of interactions between colliding pairs.

ENDNOTES

1. We note that the non-collision singularity in the collinear four-body problem, the existence of which is proved in [**3**], only occurs after the singularity of binary collisions are regularized.

2. Initially the degrees of integral are moved up by one but very soon the increment gets down to $1/2$ because of replacements that invokes (27).

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