Journal of Seismology

Estimating Ray-Based Rotational Motions from a Three Dimensional, Three Component Seismic Array

--Manuscript Draft--

Manuscript Number:	
Full Title:	Estimating Ray-Based Rotational Motions from a Three Dimensional, Three Component Seismic Array
Article Type:	Short Communication
Keywords:	rotational seismology gradiometry array processing
Corresponding Author:	Christian Poppeliers, Ph.D. Georgia Regents University Augusta, GA UNITED STATES
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	Georgia Regents University
Corresponding Author's Secondary Institution:	
First Author:	Christian Poppeliers, Ph.D.
First Author Secondary Information:	
Order of Authors:	Christian Poppeliers, Ph.D.
	Predrag Punosevac, Ph.D.
Order of Authors Secondary Information:	
Abstract:	We derive a series of equations that make it possible to estimate the rotational motions of a seismic wavefield in a ray coordinate system using a three dimensional array of three component seismic sensors. The method assumes that the recording instruments are oriented in the typical vertical, north, and east orthogonal axis. We begin by relating a general Cartesian based vector field to a vector field in spherical coordinates. Then we map Cartesian-based wavefield gradients to spherical wave gradients. The final result is a set of equations that relate the observed, Cartesian-based wavefield gradients to the ray-based rotational motions.
Suggested Reviewers:	Roman Teisseyre, Ph.D. Professor, Polish Academy of Sciences rt@igf.edu.pl expert on rotational seismology
	Eugeniusz Majewski, Ph.D. Institute of Geophysics emaj@igf.edu.pl Expert on rotational seismology
	Joachim Wasserman, Ph.D. Ludwig Maximilians Universitat joachim.wassermann@geophysik.uni-muenchen.de expert on observing rotatational seismic motions
	Nguyen Pham, Ph.D. Vietnam Academy of Science and Technology nguyen@geophysik.uni-muenchen.de Expert on observing rotational siesmic motions.

Opposed Reviewers:	

Noname manuscript No.
(will be inserted by the editor)

Estimating Ray-Based Rotational Motions from a Three Dimensional, Three Component Seismic Array

Christian Poppeliers · Predrag Punoševac

Received: date / Accepted: date

Abstract We derive a series of equations that make it possible to estimate the rotational motions of a seismic wavefield in a ray coordinate system using a three dimensional array of three component seismic sensors. The method assumes that the recording instruments are oriented in the typical vertical, north, and east orthogonal axis. We begin by relating a general Cartesian based vector field to a vector field in spherical coordinates. Then we map Cartesian-based wavefield gradients to spherical wave gradients. The final result is a set of equations that relate the observed, Cartesian-based wavefield gradients to the ray-based rotational motions.

Keywords rotational seismology · gradiometry · array processing

1 Introduction

Seismic waves are typically recorded on three-component seismometers, where the function of the instruments is to record a time record of translational ground motion at a point on the Earth's surface. However, it's well known that to completely describe the motions of a rigid body, one must also consider the angular, or rotational, motions about the three orthogonal principle axes (Lee

C. Poppeliers
Dept. of Chemistry and Physics
2500 Walton Way
Augusta, GA 30907 USA
Tel.: +1-706-667-4205
Fax: +1-706-667-4519
E-mail: cpoppeli@gru.edu

P. Punoševac
Dept. of Mathematics 2500 Walton Way
Augusta, GA 30907 USA
Tel.: +1-706-737-1672
Fax: +1-706-729-2155

et. al, 2009). Recent advances in instrumentation and seismic array techniques have made it possible to observe rotational motions, and has lead to a surge of interest in rotational seismology. Indeed, there has been been numerous recently published papers describing the theory of rotational seismic motions as well as the direct observations of rotational motions generated by both earthquakes and underground explosions (see the Bulletin of the Seismological Society of America, Vol 99, no. 2B, 2009).

Although there have been recent developments in dedicated rotational sensors (see Igel et. al, 2007), in many cases rotational motions are estimated from the translational motions recorded by seismic arrays. Regardless, in both cases the rotational motions are surface-based and observed in a surface-based Cartesian coordinate system (e.g vertical, east, and north), as it's the most natural coordinate system in terms of the instrumentation and most engineering applications. However, this observational frame is not the most natural coordinate system from the perspective of the actual wavefield nor is it ideal to observe spin and twist waves. For example, to isolate longitudinal rotational motions (i.e. torsional waves, or so-called PR waves) and shear rotation waves (termed SR waves) a ray-based coordinate system is required. PR and SR waves can be generated directly by the seismic source (Majewski, 2006) or from scattering in the heterogenious crust (Pham *et al.*, 2009; Langston et. al, 2009; Liu et al., 2009; Lin et al., 2009; Wu et al., 2009). Because translational motions associated with the interaction of a P and SV wave will generate surface tilting motions, it would be difficult to observe source generated SR wave from surface-based instrumentation. Furthermore, the interaction of a SV phase and body wave to Love wave conversion would make it difficult to directly observe source-generated PR waves.

In this letter, we develop a series of expressions that estimate the ray-based rotational motions from Cartesian-based wavefield gradients. The wavefield gradients could be estimated by, for example, a three dimensional array of multi-component instruments deployed in bore holes. This type of array deployment would be ideally isolated from the effects of the free surface. This work is an off-shoot of the theory of three dimensional wave gradiometry for polarized seismic waves (Poppeliers and Punoševac, 2012). We begin with the assumption that the seismic translational motions are known in the geographically based Cartesian coordinates. We then relate the Cartesian-based wave derivatives to the derivatives in spherical coordinates, which yields the rotational motions in ray-based spherical coordinates.

2 Statement of the Problem

In the framework of classical elasticity, and further assuming infinitesimal deformations, the seismic motions $\mathbf{u} = [u_1, u_2, u_3]^T$ are recorded in the conventional Cartesian coordinates, where the x_3 axis is perpendicular to the Earth's surface and points upwards, the x_2 axis points north, and the x_1 axis points east (Figure 1). For this definition of the coordinate axes, the rotational motions are in reference to the horizontal ground surface and are given by

$$\boldsymbol{\omega} = \frac{1}{2}\nabla \times [u_1, u_2, u_3] \tag{1}$$

where

$$\omega_1 = \frac{\partial u_3}{\partial x_2} \tag{2}$$

$$\omega_2 = -\frac{\partial u_3}{\partial x_1} \tag{3}$$

and

$$\omega_3 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \tag{4}$$

are the rotations about the x_1, x_2, and x_3 axes, respectively (Cochard et al., 2006; Lee et al., 2009).

Rotation about the vertical axis (ω_3) can result from the interaction of a longitudinal rotational wave, also termed a PR phase, with the ground surface. The motion of a PR wave is a rotation about the ray direction of the seismic wavefront, and is termed here a torsional wave (Figure 1). If the seismic ray is far from vertical, then there will necessarily be a distortion of the recorded ω_3 motion. Furthermore, coupling of the non-vertical torsional motions with the ground surface would result in rotational motions about axes that are parallel to the ground (termed tilt). To further complicate matters, tilting motions would result from the interaction of P, S with the surface, as well as the passage of surface waves.

If our goal is to observe PR and SR waves excited by the seismic source, then it is helpful if surface rotational motions resulting from the interactions of P, S, and surface waves with the surface do not contaminate our observations. Therefore, the observations must be in a ray-based coordinate system that's independent from the surface: $\mathbf{u} = [u_r, u_\phi, u_\theta]^T$ where u_r is the radial component motion and represents the direction of particle motion for P waves, u_ϕ is the transverse component motion and is the direction of particle motion for SH energy, and u_θ is the component of motion that is orthogonal to both the radial and transverse directions and is the direction of particle motion for SV energy (Figure 1).

Because the direction of the seismic ray can vary in time, the coordinate axis of the ray-based rotations can vary in time as well. Therefore, prior to estimating ray-based rotational motions, we need a method to resolve the ray direction on a point-by-point basis for the seismogram. This is possible using the method developed in Poppeliers and Punoševac (2012) or Poppeliers et al. (2012) where they resolve the time varying vector direction of the seismogram using the ratio of the horizontal and/or vertical wavefield gradients as estimated by a small-aperture, three dimensional, three component seismic array.

3 Derivation

We begin by recalling the following well known definitions. If $\mathbf{r}$ is the distance from the seismic origin to an arbitrary point and $\mathbf{r}_0$ is the center of the recording array, then the position vector of an arbitrary point in Cartesian coordinates is

$$\mathbf{r}-\mathbf{r}_0=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \tag{5}$$

where its transformation into spherical coordinates is given by

$$\begin{aligned}x_1&=(\mathbf{r}-\mathbf{r}_0)\sin\theta\sin\phi\\x_2&=(\mathbf{r}-\mathbf{r}_0)\sin\theta\cos\phi\\x_3&=(\mathbf{r}-\mathbf{r}_0)\cos\theta\end{aligned} \tag{6}$$

(Figure 1). Since we are interested in observing the rotational motions in spherical coordinates, we rewrite equation (1) as

$$\begin{aligned}\boldsymbol{\omega}&=\begin{bmatrix}\omega_r\\\omega_\phi\\\omega_\theta\end{bmatrix}\\&=\frac{1}{2}\nabla\times\begin{bmatrix}u_r\\u_\phi\\u_\theta\end{bmatrix}\\&=\frac{1}{2}\mathrm{curl}\begin{bmatrix}u_r\\u_\phi\\u_\theta\end{bmatrix}.\end{aligned} \tag{7}$$

Our ultimate goal is to derive expressions for the corresponding components of $\boldsymbol{\omega}$ in spherical coordinates, but using the Cartesian-based wavefield observations.

The first step is to find the coordinates of the vector field $\mathrm{curl}[u_r,u_\phi,u_\theta]^T$ as a functions of the coordinates of the vector field $[u_r,u_\phi,u_\theta]^T$. Given a vector field $\mathbf{A}(t)=(A_1e_1+A_2e_2+A_3e_3)(t)$ the coordinates B_1, B_2, B_3 of the field $\mathrm{curl}\mathbf{A}(t)=\mathbf{B}(t)=(B_1e_1+B_2e_2+B_3e_3)(t)$ are given as

$$\mathrm{curl}\mathbf{A}=\frac{1}{\sqrt{E_1E_2E_3}}=\begin{bmatrix}\sqrt{E_1}e_1 & \sqrt{E_2}e_2 & \sqrt{E_3}e_3\\ \dfrac{\partial}{\partial t_1} & \dfrac{\partial}{\partial t_2} & \dfrac{\partial}{\partial t_3}\\ \sqrt{E_1}A_1 & \sqrt{E_2}A_2 & \sqrt{E_3}A_3\end{bmatrix} \tag{8}$$

where $E_i,\ i=1,2,3$ are the coefficients of the matrix tensor for the coordinate system of $\mathbb{R}^3$, and $\mathbf{t}=[t_1,t_2,t_3]$ stands for a coordinate system on $\mathbb{R}^3$ with base vectors e_1,e_2, and e_3. Although it is typical in the mathematical literature to use super indices for the coordinates of the vector field following Einstein notation (e.g. $\mathbf{A}(t) = (A^1 e_1 + A^2 e_2 + A^3 e_3)$, we prefer the use of subscripts to avoid confusion with exponents.

Since our definition of spherical coordinates (6) are slightly different than normal, we compute all coefficients of (8) explicitly. We begin by computing the coefficients of the metric tensor in our spherical coordinates, which in Cartesian coordinates is given as

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2. \tag{9}$$

Note that there are diagonal coefficients in equation (9) due to the fact that our coordinate systems (Cartesian and spherical) are orthogonal. Substituting the definitions in equation (6), gives us

$$dx_1 = \sin\theta\sin\phi dr + (\mathbf{r}-\mathbf{r}_0)\cos\theta\sin\phi d\theta + (\mathbf{r}-\mathbf{r}_0)\sin\theta\cos\phi d\phi, \tag{10}$$

$$dx_2 = \sin\theta\cos\phi dr + (\mathbf{r}-\mathbf{r}_0)\cos\theta\cos\phi d\theta - (\mathbf{r}-\mathbf{r}_0)\sin\theta\sin\phi d\phi \tag{11}$$

and

$$dx_3 = \cos\theta dr - (\mathbf{r}-\mathbf{r}_0)\sin\theta d\theta. \tag{12}$$

Squaring equations 10 - 12 these gives us

$$\begin{aligned} dx_1^2 =& \sin^2\theta\sin^2\phi dr^2 + (\mathbf{r}-\mathbf{r}_0)^2\cos^2\theta\sin^2\phi d\theta^2 + (\mathbf{r}-\mathbf{r}_0)^2\sin^2\theta\cos^2\phi d\phi^2 \\ &+(\mathbf{r}-\mathbf{r}_0)\sin\theta\cos\theta\sin^2\phi \; drd\theta \\ &+(\mathbf{r}-\mathbf{r}_0)\sin^2\theta\sin\phi\cos\phi \; drd\phi \\ &+(\mathbf{r}-\mathbf{r}_0)^2\sin\theta\cos\theta\sin\phi\cos\phi \; d\theta d\phi, \end{aligned} \tag{13}$$

$$\begin{aligned} dx_2^2 =& \sin^2\theta\cos^2\phi dr^2 + (\mathbf{r}-\mathbf{r}_0)^2\cos^2\theta\cos^2\phi d\theta^2 + (\mathbf{r}-\mathbf{r}_0)^2\sin^2\theta\sin^2\phi d\phi^2 \\ &+(\mathbf{r}-\mathbf{r}_0)^2\sin\theta\cos\theta\cos^2\phi \; drd\theta \\ &-(\mathbf{r}-\mathbf{r}_0)\sin^2\theta\sin\phi\cos\phi \; drd\phi \\ &-(\mathbf{r}-\mathbf{r}_0)^2\sin\theta\cos\theta\sin\phi\cos\phi \; d\theta d\phi, \end{aligned} \tag{14}$$

and

$$dx_3^2 = \cos^2\theta dr^2 + r^2\sin^2\theta d\theta^2 - r\sin\theta\cos\theta \; drd\theta. \tag{15}$$

By adding (13), (14), and (15) we obtain a metric tensor in spherical coordinates,

$$ds^2 = dr^2 + (\mathbf{r}-\mathbf{r}_0)^2\sin^2\theta d\phi^2 + (\mathbf{r}-\mathbf{r}_0)^2 d\theta^2 \tag{16}$$

from which we compute the line element

$$ds = dr\hat{r} + (\mathbf{r}-\mathbf{r}_0)\sin\theta d\phi\hat{\phi} + (\mathbf{r}-\mathbf{r}_0)d\theta\hat{\theta}, \tag{17}$$

where $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ are the unit vectors of the moving coordinate frame:

$$\hat{r} = \frac{\dfrac{d(\mathbf{r}-\mathbf{r}_0)}{dr}}{\left|\dfrac{d(\mathbf{r}-\mathbf{r}_0)}{dr}\right|} = \begin{bmatrix} \sin\theta\sin\phi \\ \sin\theta\cos\phi \\ \cos\theta \end{bmatrix} \tag{18}$$

where

$$\left|\frac{d(\mathbf{r}-\mathbf{r}_0)}{dr}\right| = 1. \tag{19}$$

Similarly,

$$\hat{\phi} = \frac{\dfrac{d(\mathbf{r}-\mathbf{r}_0)}{d\phi}}{\left|\dfrac{d(\mathbf{r}-\mathbf{r}_0)}{d\phi}\right|} = \begin{bmatrix} \cos\phi \\ -\sin\phi \\ 0 \end{bmatrix} \tag{20}$$

where

$$\left|\frac{d(\mathbf{r}-\mathbf{r}_0)}{d\phi}\right| = (\mathbf{r}-\mathbf{r}_0)\sin\theta \tag{21}$$

and

$$\hat{\theta} = \frac{\dfrac{d(\mathbf{r}-\mathbf{r}_0)}{d\theta}}{\left|\dfrac{d(\mathbf{r}-\mathbf{r}_0)}{d\theta}\right|} = \begin{bmatrix} \cos\theta\sin\phi \\ \cos\theta\cos\phi \\ -\sin\theta \end{bmatrix} \tag{22}$$

where

$$\left|\frac{d(\mathbf{r}-\mathbf{r}_0)}{d\theta}\right| = (\mathbf{r}-\mathbf{r}_0). \tag{23}$$

We can now compute equation (7) in spherical coordinates. Let

$$\mathbf{u}(r,\phi,\theta) = u_r\hat{r} + u_\phi\hat{\phi} + u_\theta\hat{\theta} \tag{24}$$

be an arbitrary vector field in spherical coordinates. Plugging the coefficients of the spherical coordinates metric tensor (16) ($E_1 = 1$, $E_2 = (\mathbf{r}-\mathbf{r}_0)\sin\theta$, $E_3 = (\mathbf{r}-\mathbf{r}_0)^2$), the partial derivative with respect to spherical coordinates ($\frac{\partial}{\partial t_1} = \frac{\partial}{\partial r}$, $\frac{\partial}{\partial t_2} = \frac{\partial}{\partial \phi}$, $\frac{\partial}{\partial t_3} = \frac{\partial}{\partial \theta}$), and norms of the coordinate vectors in the spherical coordinates moving frame ($A_1 = 1$, $A_2 = (\mathbf{r}-\mathbf{r}_0)\sin\theta$, $A_3 = (\mathbf{r}-\mathbf{r}_0)$) into equation (8) we obtain

$$\begin{aligned} \boldsymbol{\omega} =& \frac{1}{2}\mathrm{curl}(u_r\hat{r} + u_\phi\hat{\phi} + u_\theta\hat{\theta}) \\ =& \frac{1}{2(\mathbf{r}-\mathbf{r}_0)^2\sin\theta}\begin{bmatrix} \hat{r} & (\mathbf{r}-\mathbf{r}_0)\sin\theta\hat{\phi} & (\mathbf{r}-\mathbf{r}_0)\hat{\theta} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial \theta} \\ u_r & (\mathbf{r}-\mathbf{r}_0)\sin\theta u_\phi & (\mathbf{r}-\mathbf{r}_0)u_\theta \end{bmatrix} \\ =& \frac{1}{2}\left(\omega_r\hat{r} + \omega_\phi\hat{\phi} + \omega_\theta\hat{\theta}\right) \end{aligned} \tag{25}$$

where

$$\omega_r = \frac{1}{(\mathbf{r}-\mathbf{r}_0)\sin\theta}\left(-\cos\theta\ u_\phi + u_{\theta,\phi} - \sin\theta\ u_{\phi,\theta}\right), \tag{26}$$

$$\omega_\phi = \frac{1}{(\mathbf{r}-\mathbf{r}_0)}\left(-u_\theta + u_{r,\theta} - (\mathbf{r}-\mathbf{r}_0)u_{\theta,r}\right), \tag{27}$$

and

$$\omega_\theta = \left(\frac{1}{(\mathbf{r} - \mathbf{r}_0)} u_\phi + u_{\phi,r} - \frac{1}{(\mathbf{r} - \mathbf{r}_0)\sin\theta} u_{r,\phi} \right). \tag{28}$$

Here, we use the indicial notation to denote partial derivatives. For example, $u_{i,j} = \frac{\partial u_i}{\partial x_j}$ where u_i indicates the component of u in the i direction. Equations (26-28) give us expressions for ray-based rotational motions. However it's necessary to compute the spatial derivatives of the wavefield in spherical coordinates. Because we commonly record the wavefield in Cartesian coordinates, we must determine the relationship between the spherical derivatives in equations (26-28) and the Cartesian derivatives as estimated from the Cartesian-based three dimensional array. To do so, we use ϕ, θ as Euler angles to relate a moving Cartesian frame (for example, the seismic array Cartesian coordinates) to the corresponding moving natural frame (i.e. so-called ray coordinates). The relationship between Cartesian based observations and ray-based observations is

$$\begin{bmatrix} u_r \\ u_\phi \\ u_\theta \end{bmatrix} = \mathbf{R}_\theta \mathbf{R}_\phi \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{29}$$

where $\mathbf{R}_\theta$ and $\mathbf{R}_\phi$ are rotation matrices. Explicitly, equation 29 can be written as

$$\begin{bmatrix} u_r \\ u_\phi \\ u_\theta \end{bmatrix} = \begin{bmatrix} u_1 \cos\theta \sin\phi - u_2 \cos\theta \cos\phi - u_3 \sin\theta \\ u_1 \cos\phi + u_2 \sin\phi \\ u_1 \sin\theta \sin\phi - u_2 \sin\theta \cos\phi + u_3 \cos\theta \end{bmatrix} \tag{30}$$

Poppeliers and Punoševac (2012) developed a set of equations relating the spherical derivatives to the wave's observed Cartesian derivatives:

$$\begin{bmatrix} u_{r,r} \\ u_{\phi,r} \\ u_{\theta,r} \end{bmatrix} = \mathbf{R}_\theta \mathbf{R}_\phi \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} \begin{bmatrix} \sin\phi \sin\theta \\ \cos\phi \sin\theta \\ \cos\theta \end{bmatrix}, \tag{31}$$

which can be written explicitly as

$$\begin{aligned} u_{r,r} = & \sin\theta \sin\phi(-u_{2,1} \cos\theta \cos\phi - u_{3,1} \sin\theta + u_{1,1} \cos\theta \sin\phi) \\ & + \cos\phi \sin\theta(-u_{2,2} \cos\theta \cos\phi - u_{3,2} \sin\theta + u_{1,2} \cos\theta \sin\phi) \\ & + \cos\theta(-u_{2,3} \cos\theta \cos\phi - u_{3,3} \sin\theta + u_{1,3} \cos\theta \sin\phi), \end{aligned} \tag{32}$$

$$\begin{aligned} u_{\phi,r} = & \sin\theta \sin\phi(u_{1,1} \cos\phi + u_{2,1} \sin\phi) \\ & + \cos\phi \sin\theta(u_{1,2} \cos\phi + u_{2,2} \sin\phi) \\ & + \cos\theta(u_{1,3} \cos\phi + u_{2,3} \sin\phi), \end{aligned} \tag{33}$$

and

$$\begin{aligned} u_{\theta,r} = & \sin\theta \sin\phi(u_{3,1} \cos\theta - u_{2,1} \cos\phi \sin\theta + u_{1,1} \sin\theta \sin\phi) \\ & + \cos\phi \sin\theta(u_{3,2} \cos\theta - u_{2,2} \cos\phi \sin\theta + u_{1,2} \sin\theta \sin\phi) \\ & + \cos\theta(u_{3,3} \cos\theta - u_{2,3} \cos\phi \sin\theta + u_{1,3} \sin\theta \sin\phi). \end{aligned} \tag{34}$$

Likewise, if the derivative with respect to ϕ is

$$\begin{aligned}\begin{bmatrix}u_{r,\phi}\\u_{\phi,\phi}\\u_{\theta,\phi}\end{bmatrix} =&\mathbf{R}_\theta\begin{bmatrix}\cos\phi & \sin\phi & 0\\-\sin\phi & \cos\phi & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\\&+\mathbf{R}_\theta\mathbf{R}_\phi\begin{bmatrix}u_{1,1} & u_{1,2} & u_{1,3}\\u_{2,1} & u_{2,2} & u_{2,3}\\u_{3,1} & u_{3,2} & u_{3,3}\end{bmatrix}\begin{bmatrix}r\cos\phi\sin\theta\\-r\sin\phi\sin\theta\\0\end{bmatrix}\end{aligned} \tag{35}$$

then

$$\begin{aligned}u_{r,\phi} =& \cos\theta\cos\phi((\mathbf{r}-\mathbf{r}_0)\sin\theta(u_{1,1}+u_{2,2})\sin\phi+u_1)\\&+\cos\theta\sin\phi(u_2-(\mathbf{r}-\mathbf{r}_0)u_{1,2}\sin\theta\sin\phi)\\&-(\mathbf{r}-\mathbf{r}_0)u_{2,1}\cos\theta\sin\theta\cos^2\phi\\&+(\mathbf{r}-\mathbf{r}_0)\sin^2\theta(u_{3,2}\sin\phi-u_{3,1}\cos\phi),\end{aligned} \tag{36}$$

$$\begin{aligned}u_{\phi,\phi} =& -\sin\phi((\mathbf{r}-\mathbf{r}_0)u_{2,2}\sin\theta\sin\phi+u_1)\\&+(\mathbf{r}-\mathbf{r}_0)u_{1,1}\sin\theta\cos^2\phi\\&+\cos\phi\left(u_2-(\mathbf{r}-\mathbf{r}_0)\sin\theta(u_{1,2}-u_{2,1})\sin\phi\right),\end{aligned} \tag{37}$$

and

$$\begin{aligned}u_{\theta,\phi} =& \sin\theta\left(\cos\phi((\mathbf{r}-\mathbf{r}_0)\sin\theta(u_{1,1}+u_{2,2})\sin\phi+(\mathbf{r}-\mathbf{r}_0)u_{3,1}\cos\theta+u_1)\right)\\&+\sin\theta\left(\sin\phi(-(\mathbf{r}-\mathbf{r}_0)u_{1,2}\sin\theta\sin\phi-(\mathbf{r}-\mathbf{r}_0)u_{3,2}\cos\theta+u_2)\right)\\&-(\mathbf{r}-\mathbf{r}_0)u_{2,1}\sin^2\theta\cos^2\phi.\end{aligned} \tag{38}$$

Finally, because

$$\begin{aligned}\begin{bmatrix}u_{r,\theta}\\u_{\phi,\theta}\\u_{\theta,\theta}\end{bmatrix} =&\begin{bmatrix}-\sin\theta & 0 & -\cos\theta\\0 & 0 & 0\\\cos\theta & 0 & -\sin\theta\end{bmatrix}\mathbf{R}_\phi\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\\&+\mathbf{R}_\theta\mathbf{R}_\phi\begin{bmatrix}u_{1,1} & u_{1,2} & u_{1,3}\\u_{2,1} & u_{2,2} & u_{2,3}\\u_{3,1} & u_{3,2} & u_{3,3}\end{bmatrix}\begin{bmatrix}r\sin\phi\cos\theta\\r\cos\phi\cos\theta\\-r\sin\theta\end{bmatrix}\end{aligned} \tag{39}$$

then

$$\begin{aligned}u_{r,\theta} =& (\mathbf{r}-\mathbf{r}_0)\cos^2\theta\left(u_{1,1}\sin^2\phi+(u_{1,2}-u_{2,1})\sin\phi\cos\phi-u_{2,2}\cos^2\phi\right)\\&-\cos\theta(\sin\theta\sin\phi((\mathbf{r}-\mathbf{r}_0)u_{3,1}+u_{1,3})-\sin\theta\cos\phi(u_{2,3}-ru_{3,2})+u_3)\\&+\sin\theta(-u_1\sin\phi+u_2\cos\phi+u_{3,3}\sin\theta),\end{aligned} \tag{40}$$

$$\begin{aligned}u_{\phi,\theta} =& \cos\phi((\mathbf{r}-\mathbf{r}_0)\cos\theta(u_{1,1}+u_{2,2})\sin\phi+u_1-u_{1,3}\sin\theta)\\&+(\mathbf{r}-\mathbf{r}_0)u_{1,2}\cos\theta\cos^2\phi\\&+\sin\phi((\mathbf{r}-\mathbf{r}_0)u_{2,1}\cos\theta\sin\phi+u_2-u_{2,3}\sin\theta),\end{aligned} \tag{41}$$

and

$$
\begin{aligned}
u_{\theta,\theta} =& -\cos\theta\sin\theta\left(u_{3,3} - (\mathbf{r}-\mathbf{r}_0)u_{1,1}\sin^2\phi\right) \\
& + \cos\theta\cos\phi(u_2 - (\mathbf{r}-\mathbf{r}_0)\sin\theta(u_{1,2}-u_{2,1})\sin\phi) \\
& + \cos\theta\left((\mathbf{r}-\mathbf{r}_0)u_{2,2}\sin\theta\cos^2\phi - u_1\sin\phi\right) \\
& + (\mathbf{r}-\mathbf{r}_0)\cos^2\theta(u_{3,1}\sin\phi + u_{3,2}\cos\phi) + \sin\theta(-u_{1,3}\sin\theta\sin\phi \\
& \qquad + u_{2,3}\sin\theta\cos\phi - u_3).
\end{aligned} \tag{42}
$$

Finally, combining the equations for the radial derivatives (equations 32-34, 36-38, 40-42) into the definitions for the radial rotations (equations 26-28) gives us expressions for the rotational motions in spherical coordinates from Cartesian-based observations:

$$
\begin{aligned}
\omega_r =& \frac{1}{(\mathbf{r}-\mathbf{r}_0)}\Big[-\cos\phi\Big(\sin\theta((\mathbf{r}-\mathbf{r}_0)(u_{1,1}+u_{2,2})\sin\phi + u_{1,3}) - u_1\cot\theta\Big) \\
& + \frac{1}{2}(\mathbf{r}-\mathbf{r}_0)\cos\theta\Big(u_{1,1}\sin 2\phi + (u_{1,2}-u_{2,1})\cos 2\phi + u_{1,2} + u_{2,1} \\
& \qquad + u_{2,2}\sin 2\phi - 2u_{3,1}\cos\phi + 2u_{3,2}\sin\phi\Big) \\
& + \sin\phi\Big((\sin\theta((\mathbf{r}-\mathbf{r}_0)u_{1,2}\sin\phi - u_{2,3}) + u_2\cot\theta\Big) \\
& + (\mathbf{r}-\mathbf{r}_0)u_{2,1}\sin\theta\cos^2\phi\Big],
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\omega_\phi =& \frac{1}{(\mathbf{r}-\mathbf{r}_0)}\Big[\sin\theta\Big(\sin\theta\left(u_{3,3} - (\mathbf{r}-\mathbf{r}_0)u_{1,1}\sin^2\phi\right) \\
& \qquad + \cos\phi(2u_2 - (\mathbf{r}-\mathbf{r}_0)\sin\theta(u_{1,2}-u_{2,1})\sin\phi) \\
& \qquad + (\mathbf{r}-\mathbf{r}_0)u_{2,2}\sin\theta\cos^2\phi - 2u_1\sin\phi\Big) \\
& - (\mathbf{r}-\mathbf{r}_0)\cos^2\theta(-u_{1,1}\sin^2\phi - (u_{1,2}-u_{2,1})\sin\phi\cos\phi \\
& \qquad + u_{2,2}\cos^2\phi + u_{3,3}) \\
& - \cos\theta(\sin\theta\sin\phi((\mathbf{r}-\mathbf{r}_0)u_{1,3} + 2(\mathbf{r}-\mathbf{r}_0)u_{3,1} + u_{1,3}) \\
& - \sin\theta\cos\phi((\mathbf{r}-\mathbf{r}_0)u_{2,3} - 2(\mathbf{r}-\mathbf{r}_0)u_{3,2} + u_{2,3}) + 2u_3)\Big].
\end{aligned} \tag{44}
$$

and

$$
\begin{aligned}
\omega_\theta =& \frac{1}{(\mathbf{r}-\mathbf{r}_0)}\Big[\cos\phi\Big((\mathbf{r}-\mathbf{r}_0)u_{1,1}\sin\theta\sin\phi + (\mathbf{r}-\mathbf{r}_0)\cos\theta(u_{1,3} - (u_{1,1}+u_{2,2})\sin\phi) \\
& \qquad + (\mathbf{r}-\mathbf{r}_0)u_{2,2}\sin\theta\sin\phi + (\mathbf{r}-\mathbf{r}_0)u_{3,1}\sin\theta - u_1\cot\theta + u_1\Big) \\
& + \sin\phi\Big((\mathbf{r}-\mathbf{r}_0)\cos\theta(u_{1,2}\sin\phi + u_{2,3}) + (\mathbf{r}-\mathbf{r}_0)u_{2,1}\sin\theta\sin\phi \\
& \qquad - (\mathbf{r}-\mathbf{r}_0)u_{3,2}\sin\theta - u_2\cot\theta + u_2\Big) \\
& + (\mathbf{r}-\mathbf{r}_0)\cos^2\phi\Big(u_{1,2}\sin\theta + u_{2,1}\cos\theta\Big)\Big].
\end{aligned} \tag{45}
$$

where the angles ϕ and θ may be time varying.

4 Discussion and Conclusions

The main purpose of the work presented here is to develop a set of expressions that can estimate the time varying, ray-based rotational motions based on Cartesian observations of body wave seismograms. The analysis relies on a set of Cartesian-based based wavefield spatial gradients which could be estimated from a three-dimensional, three-component seismic array (Poppeliers *et al.*, 2012; Poppeliers and Punoševac, 2012). The primary goal of observing seismic rotational motions in this way is that it would be possible to observe source generated PR and SR waves in the first arriving seismic energy (e.g. Takeo and Ito, 1997), isolated from the effects of the free surface.

Surface observations of PR and SR motions would be extremely difficult to resolve due to contamination of rotational motions introduced by the effects of the free surface. Specifically, the P and SV waves interacting with the surface will induce tilting motions, and the interaction of the SH wave with the free surface will cause rotions about the vertical axis. These motions will be superposed with true PR and SR waves generated by the seismic source or from seismic scattering (Pham *et al.*, 2009). Therefore, an array that is isolated from the free surface would be the best candidate to observe source-generated rotational motions. However, given that resolving PR and SR phases depends on the wavefield direction by a stationary array, we employ some techniques that were developed in Poppeliers *et al.* (2012) and Poppeliers and Punoševac (2012). Specifically, they developed a mapping between Cartesian-based observations, the wavefield direction (as could be resolved from 3-D gradiometric analysis), and ray-based spatial derivatives. This mapping leads directly to the results presented here. Although a buried, three-dimensional, three-component seismic array has not yet been deployed, given the small aperture required for this type of analysis we argue that such a deployment is technically feasible via borehole instrumentation.

Clearly the technical aspects of such an array would need to be resolved. For example, we anticipate that the array would need to be several tens of wavelengths below the surface to eliminate surface effects. Furthermore, what is the minimum depth, and how does this depth depend on the frequency of the observed wavefield, its speed, and the surrounding geologic material influence the burial depth? Rigorous testing, both with synthetic seismograms and with an actual deployment, would need to occur before these technical aspects could be resolved.

The main application to the methods we develop here are to resolve the nature of the seismic source. For example, Poppeliers *et al.* (2012) and Poppeliers and Punoševac (2012), developed a series of partial differential equations that relate the spatial gradients of the wavefield with the radiation patterns. The work here would provide additional information on the rotational motions generated by the seismic source, which could further help to characterize the siemsic source. Such information would help in characterizing earthquake faults, but could also be used in an industrial setting. Specifically, small, high frequency gradiometers could be used to help characterize the fracture mechanisms caused by induced hydraulic fracturing during petroleum extraction operations.

Acknowledgements Acknowledge reviewers, when known.

References

1. Cochard, A., H. Igel, B. Schuberth, W. Suryanto, A. Velikoseltsev, U. Schreiber, J. Wassermann, F. Sherbaum, and D. Volmer, Rotation motions in seismology: Theory, observation, simulation, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, R. Teisseyre, M. Takeo, and E. Majewski (Editors). 391-441. Springer-Verlag, Heidelberg (2009).
2. Igel, H., A. Cochard, J. Wasserman, A. Flaws, U. Schreiber, A. Velikoseltsev, and N. P. Dinh (2007). Broad-band observations of earthquake-induced rotational ground motions, *Geophys. J. Int.*, **168**, 182-196 (2009)
3. Langston, C.A., W.H. K. Lee, C.J. Lin, and C.C. Lui, Seismic wae strain, rotation, and gradiometry for the 4 March 2008 TAIGER explosions, *Bull. Seis. Soc. Am.* **99**, no. 2B, 1287-1301 (2009)
4. Lee, W. H. K., B. -S. Huang, C. A. Langston, C. -J. Lin, C. -C. Liu, T. -L. Shin, T. Teng, and C. -F. Wu, Review: Progress in Rotatoinal Ground-Motion Observations from Explosions and Local Earthquakes in Taiwan, *Bull. Seis. Soc. Am.*, **99**, no. 2B, 958-967 (2009)
5. Lin, C. -J., C. -C. Liu, and W. H. K. Lee, Observing rotational and translational ground motions of two TAIGER explosions in northeastern Taiwan on 4 March 2008, *Bull. Seis. Soc. Am.*, **99**, no. 2B, 1237-1250 (2009)
6. Liu, C.-C., C.-S. Huang, W.H.K. Lee, and C. -J. Lin, Observing roational and translational ground motions at the HGSD station in Taiwan from 2007 to 2008, *Bull. Seis. Soc. Am.*, **99**, no. 2B, 1228-1236 (2009)
7. Majewski, E., Seismic Rotation Waves: Spin and Twist Solitons, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, R. Teisseyre, M. Takeo, and E. Majewski (Editors). Springer-Verlag, Heidelberg, 255-272 (2006)
8. Pham, N.D., H. Igel, J. Wasserman, M. Kȧser, J. de la Puente, and U. Schreiber, Observations and Modeling of Rotational Signals in the P Coda: Constraints on Crustal Scattering, *Bull. Seis. Soc. Am.*, **99**, no. 2b, 1315-1332 (2009)
9. Poppeliers, C., P. Punoševac, Three dimensional wave gradiometry for polarized seismic waves, *Bull. Seis. Soc. Am.*, in review (2012)
10. Poppeliers, C., P. Punoševac, and T. Bell, Three dimensional seismic wave gradiometry for scalar waves, *Bull. Seis. Soc. Am.*, in review (2012)
11. Takeo, M., H, M. Ito, What can be learned from rotational motions excited by earthquakes? *Geophys. J. Int.*, **129**, 319-329 (1997)
12. Wu, C. -F., W. H. K. Lee, and H. -C Huang, Array deployment to observe rotational and translational ground motions along the Meishan fault, Taiwan: a progress report, *Bull. Seis. Soc. Am.*, **99**, no. 2B, 1468-1474 (2009)

line figure
Click here to download line figure: fig1.eps