Math 3020 Differential Equations
Section A

Instructor: Dr. Predrag Punoševac

Final Exam

Student Name:__________________
Student ID#:__________________

Each problem is worth 10 points. Give a complete solution to receive the full credit!

---

1. Duffing [1918] introduced a nonlinear oscillator with the cubic stiffness term and the friction depending on parameter δ to describe the hardening spring effect observed in many mechanical problems.

$$\ddot{x} + \delta\dot{x} - x + x^3 = 0 \tag{1}$$

State whether the Duffing's equation (1) is autonomous or non-autonomous ordinary differential equation (ODE for short). Determine whether it is linear or nonlinear ODE. What is it order?

2. We introduce an additional parameter β which measures the relative strength of the elastic forces.

$$\ddot{x} + \delta\dot{x} - \beta x + x^3 = 0 \tag{2}$$

Use the substitution

$$x_1 \stackrel{\text{def}}{=} x$$
$$x_2 \stackrel{\text{def}}{=} \dot{x}$$

to write the equation (2) as the system of two first order ODEs.

3. Let us recall the definition of *equilibrium position*.

Definition 1 *A point $\vec{x}_0$ is called an equilibrium position of the system*

$$\dot{\vec{x}} = \vec{f}(\vec{x}), \quad \vec{x} \in \mathbb{R}^n \tag{3}$$

if $\vec{x}(t) \equiv \vec{x}_0$ is the solution of the system (3). In other words, $\vec{f}(\vec{x}_0) = 0$.

Suppose that $\beta < 0$. How many *equilibrium positions* does the system obtained from the equation (2) have.

4. We now study an autonomous system

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ \beta & -\delta \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -{x_1}^3 \end{pmatrix} \tag{4}$$

where parameters $\delta > 0$ and $\beta > 0$. The system (4) under the assumptions about parameters has three *equilibrium positions* $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (\pm\sqrt{\beta}, 0)$.

In studying solutions of (4) which are close to equilibrium position $(x_1, x_2) = (0, 0)$, we often use *linearization* i.e. we pass from the system (4) to the system

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ \beta & -\delta \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{5}$$

Compute the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ \beta & -\delta \end{bmatrix}.$$

5. Find the general solution of the system (5).

6. Recall

Theorem 1 *(Hartman-Grobman) If A (as in problem 4) has no zero or purely imaginary eigenvalues then there is a homeomorphism h defined in some neighborhood U of $(x_1, x_2) = (0, 0)$ in $\mathbb{R}^2$ locally taking orbits of the nonlinear flow ϕ_t of (4) to those of the linear flow e^{At}.*

Does the system (4) qualitatively behave the same as the system (5). Justify your claim!

7. Suppose now that $\beta = 1$ and $\delta = 0$ so that the system (5) becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{6}$$

Draw the phase portrait of the system (6).

Hint: Note that the phase curves of the system (6) are integral curves of the equation

$$\frac{dx_1}{dx_2} = \frac{x_2}{x_1}. \tag{7}$$

8. Moon and Holmes [1979,1980] showed that the Duffing equation in the form

$$\ddot{x} + \delta\dot{x} - x + x^3 = \gamma \cos \omega t \tag{8}$$

provided the simplest possible model for the forced vibrations of a cantilever beam in the nonuniform field of the permanent magnets. The analysis of (9) exceed the scope of this course. However you should be able to write a general solution of the equation

$$\ddot{x} + \delta\dot{x} - x = \gamma \cos \omega t. \tag{9}$$

Assume further $\delta = \gamma = \omega = 1$ and write a general solution to the equation (9).

9. Two students are memorizing a list of chemical compositions. Their memorization rates are both given by the differential equation

$$\frac{dL}{dt} = 3(1 - L)$$

One student started yesterday and has memorized half of the list and the other is starting today and has memorized none of it. Will the student that is starting today ever catch up with the one that started yesterday? Carefully justify your answer.

Hint: Recall the theorem on existence and uniqueness of solutions to first-order ODE.

10. Write in words the following statement:

$$\neg(Mathematics \subseteq Physics).$$