Undergraduate Research in Mathematics

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This document is written in an attempt to help mathematics faculty develop or expend upon their current mentorship and encourage them to involve undergraduate students in mathematics research. It is based on my personal experience in trying to do research with my own mentees as well as the input from my U.S. colleagues who have a track record of supervising successful undergraduate research

1 Introduction

1.1 A Brief History of Undergraduate Research

The year 2012 marks the 25th anniversary of the first NSF REU program. Such a short history comes a bit as a surprise as most European universities include in their graduation requirements mandatory final diploma work which bares all the characteristics of REU. It is worth noting a change in U.S. mathematics culture over this period for which REUs have been the catalyst. The language of nature is mathematics! It is a very simple language but you have to be a genius to understand it. Not long ago, most mathematicians in U.S. believed that undergraduates simply were not able to perform professional quality research in mathematics. However, the REU programs have helped to change the minds of many. What is the evidence?

- At the January joint AMS-MAA mathematics meeting in 1991 there were 12 exhibits at the poster session on research by undergraduates. At the same session in 2012 there were 310 posters representing the research of 525 students.
- At the 1993 joint meetings, the first at which the number of undergraduates were tracked, 71 were registered. At the 2012 joint meeting 948 undergraduates were registered.
- In 1995 15 NSF REU supplements in mathematics were active, in 2012 89 were active.
- At the 1996 joint meetings, the first at which the number of talks by undergraduates was tracked, there were six. At the 2012 joint meeting, 152 undergraduates gave talks.
- Solo authored papers by undergraduates are being published in leading professional journals: *Annals of Mathematics*, *Annals of Probability*, *Journal fur die reine und angewandte Mathematik*, *Journal of Algebra* and many others.

1.2 What is Mathematical Research?

Mathematics is a very old and important part of human culture. We find its origins in the human attempts to quantify the laws of nature. It is tempting to think of modern mathematics as a cult of truth, founded by Isaac Newton three hundred years ago. Whatever your views of this subject, it is easy to recognize some of its features: abstractness, precision, rigor, indisputable character of its conclusions, and finally, the exceptionally broad range of its applications.

One of the most common questions I am asked by people when they learn that I am a research mathematician is ”What do people research in Mathematics?”. They usually go on to state that in mathematics everything is already known since we know that two plus two is four. Much like in any other discipline **mathematical research** is work which is undertaken systematically to increase the stock of knowledge by discovering new theorems or proving existing hypothesis, reaffirm the results of previous work, solve new or existing problems, or develop new theories.

The principal reason that people and students in particular have such a hard time contemplating that somebody can discover something new in mathematics is that they knowledge limits their ability to ask really hard open question. Unlike in disciplines like Astronomy where even major discoveries are made by amateurs, mathematics require substantial familiarity with the existing body of knowledge before the research can take the place.

Unlike a biology research project that might help discovering a critically-needed vaccine the implications of theoretical mathematical research on our every-day's life appears to be less obvious. Nothing could be further from the truth as every time you use your cell phone, turn on your mp3 player, watch a movie, pull your credit card to pay a bill or even get a flu vaccine you are just enjoying the fruits of theoretical mathematical research.

1.3 On Teaching Mathematics

Teaching an advanced undergraduate or graduate course has a dual purpose. On the one hand, there is a certain body of advanced material to be presented and learned by the members of the audience. The students should be able to reproduce the proofs and apply results in solving problems. On the other hand, such a course is an opportunity to teach students certain ways of thinking in mathematics where specific material is used to illustrate mathematical ideas.

My favored approach is to introduce empirical evidence first and reinforce the point of view that mathematics is the part of physics where experiments are cheap. I tend to skip even essential details at first, and try to bring students' attention to the structural and logical elements which will play key roles in the development of the particular mathematical result I am trying to present. However, as observed by Courant, empirical evidence alone can never establish mathematical existence of the real world phenomena. Only a mathematical existence proof can ensure that the mathematical description of a physical phenomenon is meaningful. Therefore, I follow up by providing rigorous mathematical proofs. I do not like using the most elegant arguments but prefer instead the arguments which will be most instrumental in exposing the key ideas of the proof. I am not a fan of well-polished results typically found in the good textbooks. I prefer to dissect statements of theorems to the simplest and the most specific cases in such a way that no further simplifications are possible, while still preserving the content of the original mathematical assertions. I love to reveal the motivation that led to the discovery and supply computational examples whenever possible. I am a fan of the guided discovery method, better known as the Socratic method.

2 The Role of Undergraduate Research

The most noticeable characteristics of undergraduate research in mathematics are:

2.1 Working on the Non-trivial Open Problem

Well posed problems with known answers are very exceptional in mathematics. Most research questions are open ended, often with out clear formulation and even less predictable answers. So it seems counter intuitive that we use exactly the opposite kind a questions to train people. While conducting undergraduate research students are asked to work on the non-trivial (means require mental process beyond couple of hours or days) open (in the sense that problems are open for students not necessary really open research questions).

2.2 Changing a Nature of Traditional Classroom

Traditional class period classroom model is confrontational in its nature. Students are on one side while professor is on the opposite side. One side seems to know everything while the other knows nothing. The punishments and rewords are always single directional. This is again completely in contradiction with the nature of human work and mathematics work in particular. The nature of mathematics work with very rare exception is collaborative. We talk to our colleagues, we read other people's papers, we listen lectures. We eagerly expect approval of our pears and fear their disapproval of our own mathematical ideas. So why then we use the classroom? Because it is a cheap and streamlined way to transfer some knowledge to our students. While doing undergraduate research students are engaged in the environment with most fatefully resembles the environment in which they are likely to work after graduation. The ideas are flowing bi-directional and the work is as rewording for advisors as for the students.

2.3 Active Learning Environment

Traditional class-period classroom model is the one in which students are rather passive observers than active participants. As we all know active learning is the only way to learn mathematics. One can only learn mathematics by doing it.

2.4 Vertically Integrated Professional Development

While doing undergraduate research students quickly learn to use research literature independently and utilize library facilities. They quickly become proficient in TeX and other tools of the trade. Finally, they learn how to conduct themselves professionally and establish and maintain professional relations with collaborators.

2.5 Other Benefits

What are some other benefits of undergraduate research in mathematics? Students who engage in research

- gain self confidence,
- improve their writing and speaking skills,
- learn teamwork,
- develop an understanding of what research mathematicians do,
- receive and introduction to the profession,
- are more likely to enter graduate school,
- are more likely to get into better graduate school,
- are more likely to do better research in graduate school,
- are more likely to get a Ph.D.,
- are more likely to get a better job offer, and
- are better prepared to do projects required by employers.

2.6 Final Remarks on the Role of Undergraduate Research

So is undergraduate research a real research? Can undergraduate students obtain a genuine research results in mathematics? As noted in introduction undergraduate research can be a real research and some undergraduate students can obtain a genuine research results. It might come as a surprise that some of the most successful RUE programs in the country which have produced world's best undergraduate mathematics research have not been lead by world's top research mathematicians and hosted at Ive league schools. On another some undergrad places are surprisingly delusional about the "research" that their REU programs supposedly support.

3 Undergraduate Math Research at GRU

Upon my arrival to former Augusta State University, I took a formidable task of trying to engage undergraduate students in mathematics research. My previous experience consisted of mentoring a small team of students during the 2007 Arizona Summer Program on Mathematical Modeling, an NSF sponsored 4-week research experience for undergraduates at the University of Arizona. Most participants in Arizona program were graduating seniors who were already admitted into graduate programs or were contemplating going to graduate school. As my audience in Augusta was very different I decided to have another approach.

3.1 Recruitment and Eligibility

It is very important for any advisor engaged in undergraduate research and undergraduate research in mathematics in particular, not to underestimate the potential for original undergraduate research. It is all too easy to create failure by expecting failure than it is to give students an exaggerated view of their own abilities. Mathematics research is difficult, but there's no reason to make it even more difficult by creating an atmosphere where failure is the expected norm. In my point of view ability to think clearly and originally is far more important in doing research than factual knowledge and familiarity with current literature. My selection process is long and starts of early in the Fall when I ask interested students to sign for Putnam Mathematical Competition. The Putnam competition is an annual mathematics competition for undergraduate college students of the United States and Canada which is held on the first Saturday in December. The problems are very deep and require ingenuity rather then applications of learned facts. During the Putnam preparation season preceding the competition I meet with students once a week for two hours two work on problems. We usually sit quietly together around the table and work individually until somebody makes a progress on a problem. At that point the person who thinks that she/he made a progress tries to convince the other team members in correctness of her/his solution and write up the proof. It should come as a no surprise that I am not always the first one who makes a progress. Seeing me struggling with problems is usually very humbling experience for students. Sometimes our sessions end up as a lectures initiated by students question about unfamiliar notions in their attempt to understand a question.

The attrition rate during the preparation season is very high and taking a Putnam test in December is usually a very strong indication of exceptional mathematics aptitude if not ability.

3.2 How is it done?

The actual research starts in the spring semester following Putnam preparation season. It starts by me asking students to read a recent research article and propose a question worth investigating. The amount of knowledge required to understand recent progress usually far exceeds student familiarity with the subject and requires from them to become squinted with new mathematical ideas on their own in very rapid fashion. It is of paramount importance at this stage not to forget that the goal of doing mathematics research is not learning the subject but rather discovering new things. This puts a great responsibility on the mentor on one hand in assigning student doable problems and on the another hand preventing them from just studying. It is often the case that the problems I assign gives students lead to no results. Sometimes after a few weeks of hard work, a student makes no significant progress. In these situations I give a new problem to the student. Matching students and problems is a tricky process that requires a lot of educated guessing tempered by experience which I have not quite mastered yet. Finding suitable problems is even greater challenge and I am on the constant lookout while reading peer-reviewed Journals, talking to experts in the field and leaders of better known REU programs. A creating a local data base of suitable papers/problems would be a great addition to my current GRU effort.

It is absolutely mandatory that students write their work in a format suitable for publication in a research journal! If you don't write up your results, it's just as if you didn't even get them at all. Writing for publication is an important aspect of being a professional mathematician. The publication process can be a very intimidating experience, even for experienced mathematicians. Without help, most students would not even realize that their work is of professional caliber.

At the end of spring program the students are also required to give a presentation on their research to other interested students and faculty.

3.3 Project: Numerical investigation of strange attractors

Following [2] students in this project will study a dissipative standard map-like system which generalizes the (conservative) Chirikov standard map to dissipative case as described by the equations

$$\begin{aligned}\bar{y} &= by + c + \frac{\epsilon}{2\pi}\sin(2\pi x)\\ \bar{x} &= x + \bar{y}\end{aligned} \tag{1}$$

where $y \in \mathbb{R}$, $x \in [0.1)$, $c \in \mathbb{R}$, $b \in [0, 1]$, $\epsilon \in \mathbb{R}_+$. Notice that the determinant of the Jacobian of (1) is equal to b. In particular, the mapping is conservative whenever $b = 1$ and it is integrable whenever $\epsilon = 0$ i.e. the mapping is governed by two parameters measuring the strength of the dissipation and of the perturbation. It is well known that for certain values of parameters Chirikov standard map bifurcates from quasi-periodic behavior studied by the means of the Kolmogorov-Arnold-Moser (KAM) theorem illustrated in the next figure by plotting Poincare section on a KAM torus with a simple Python code.

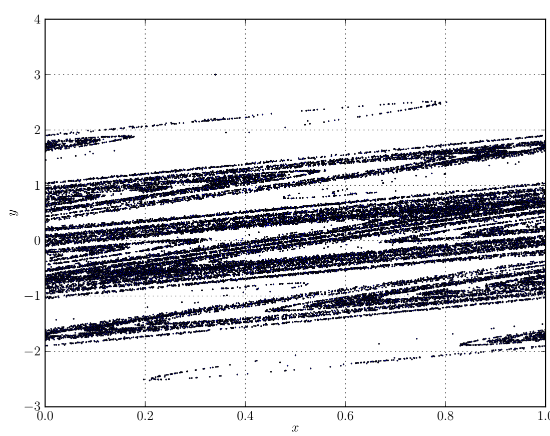
```
import numpy as np
import scipy as sp
import matplotlib as mpl
import matplotlib.pyplot as plt
import math

mpl.rc('text',usetex=True)
mpl.rc('font',**{'family':'serif','serif':['Computer Modern']})
mpl.rc('font', size=10.0)
mpl.rc('legend', fontsize=10.0)
mpl. rc('font', weight='normal')
```

```
def gen_chirikov(x,y):
    alpha = (pow(5,0.5)-1.0)/2.0
    b = 0.72699
    c = 0
    eps = 0.971635*2*np.pi
    yy = b*y+c+ eps/(2.*np.pi)*np.sin(2*np.pi*x)
    xx = x + yy
    return(xx,yy)

file = open("out.dat", 'w')
for x in np.linspace(0, 1, 20, endpoint=True):
    for y in np.linspace(0, 10, 20, endpoint=True):
        for i in range(1,6100):
            (x,y)=gen_chirikov(x,y)
            if i > 6000 :
                file.write(str(x-math.floor(x))+' '+str(y)+'\n')
file.close()


data = np.genfromtxt('out.dat')
x = data[:,0]
y = data[:,1]
plt.plot(x,y,'o',markersize=1)
plt.xlabel(r'$x$')
plt.ylabel(r'$y$')
plt.grid(True)
plt.figure(figsize=(4, 2.5))
plt.legend(loc='lower right')
plt.savefig('myplot.pdf', bbox_inches='tight')
```



One of the goals of this project is to obtain more information about dissipative standard map-like. The work on this project will involve both computer experiments and theoretical study, but individual students might put more emphasis on one or other. We expect students to be involved in all stages of the process: planning the study, doing the programming for the computations, and interpreting the results as well as trying to prove the conjectures that come out of the process. Students are also expected by the end of the semester to write a paper summarizing the results of their investigation in the form appropriate for submission to a pear-reviewed journal. Students are also expected to give a conference talk reporting the progress they made.

3.4 Final Outcome

Publishing a paper should not be a final outcome for students engaged in undergraduate research. My students are very strongly encouraged during the spring semester to apply for notable (MASS, Cornell) REU programs as well as study abroad such as Math in Moscow and Budapest Semester in Math. Participating and giving talks at MAA sectional meetings and Yount Mathematicians Conference. Students who wish further to pursue graduate studies in Mathematics are matched with potential graduate advisors, mentored through the application process and encouraged to attend recruitment workshops.

4 Where to go from here?

Instead of conclusion in this section we will list some of the resources available to the faculty who are interested in supervising undergraduate research.

4.1 Some Successful Research Projects

It never hurts to compare with fellows from other colleges. This is how successful research projects look at Cornell University.

4.2 Undergraduate Research Journals

New discoveries in mathematics are unlikely events. For the most part we are ignorant of the nature of these events and of their probabilities. Some of them are at present quite beyond our control. It is unlikely that a typical undergraduate student will be able to produce work worth considering by a professional mathematical journal. Non the less it is absolutely mandatory that students write their work in a format suitable for publication in a research journal! There are several journal specialized in publishing such papers:

- SUIRO
- Rose-Hulman Undergraduate Mathematics Journal
- The Furman University Electronic Journal of Undergraduate Math

4.3 Undergraduate Mathematics Conference

It is a growing trend that many regional conferences offer 15 minutes time slots to undergraduates to present they finding. However there is increasing number of conferences totally dedicated to this kind presentations:

- MAA Regional Undergraduate Mathematics Conferences
- The Young Mathematicians Conference

4.4 Notable REU Programs

The list of all REU programs is maintained by American Mathematical Society. However those run by Penn State and Cornell are in particularly recognized for the highest quality standards. Applicants should note that most application deadlines fall in February - March for summer programs. Financial support is usually provided by organizers or directly from NSF.

4.5 Study Abroad Programs

The study abroad programs are crown jewels in AMS outreach program. Two most important once are:

- Math in Moscow
- Budapest Semester in Mathematics

The application process is very competitive and AMS has several scholarship programs.

4.6 Recruitment Workshops

Many graduate programs in the country as a part of their recruitment activities have week long workshops. I was personally involved several years in the one offered by the University of Arizona.

References

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[4] Susan H Russell, Mary P Hancock, and James McCullough, *Benefits of undergraduate research experiences*, Science(Washington) **316** (2007), no. 5824, 548–549.